Asymptotic of twisted Alexander polynomials and hyperbolic volume

L. Benard, J. Dubois, M. Heusener, J. Porti

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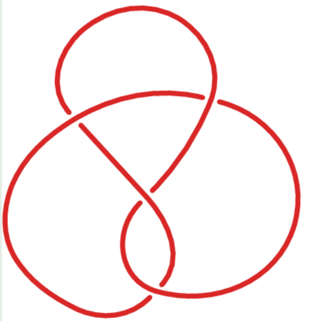
which acts on $\mathbb{H}^3$ by isometries.

Hyperbolic manifolds

We consider M a hyperbolic 3-manifold of finite volume, for instance

Example (The figure-eight knot)



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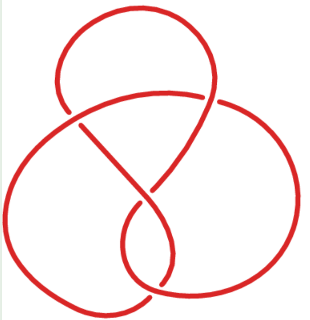

It follows from Mostow rigidity that the volume of M is a topological invariant.

Reidemeister torsion

An other topological invariant is the Reidemeister torsion of the pair

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It should be thought as the alternating product of the "determinants" of the boundary operators d_i:

$$\mathrm{tor}(M,\rho) = \prod_i \det(d_i)^{(-1)^i} \in \mathbb{C}^*$$

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For instance

$$\mathrm{Sym}_{n-1}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^{n-1} & & & & \\ & \lambda^{n-3} & & & \\ & & \ddots & & \\ & & & \lambda^{3-n} & \\ & & & & \lambda^{1-n} \end{pmatrix}$$

Asymptotic of torsions

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Previous question has been answered positively as follows:

Theorem (Müller '12 for the compact case, Menal-Ferrer–Porti '14 for the general case)

Denote by

$$\rho_n \colon \pi_1(M) \xrightarrow{\rho} \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\mathrm{Sym}_{n-1}} \mathrm{SL}_n(\mathbb{C})$$

the $(n-1)$ symmetric power of the holonomy representation of a hyperbolic manifold M. The following holds:

$$\lim_{n\to\infty} \frac{\log|\mathrm{tor}(M,\rho_n)|}{n^2} = \frac{\mathrm{Vol}(M)}{4\pi}$$

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First recall that the holonomy representation ρ is part of a moduli space of deformations of geometric structures, the deformation variety, or character variety.
In a few words, despite the holonomy representation corresponds to the unique (thanks to Mostow rigidity) complete hyperbolic structure on M, one can deform this structure into non-complete ones, yielding a moduli space.
Moreover, this character variety is an analytic (even algebraic) variety, equipped with analytic functions

$$\mathrm{tor}\colon [\varrho] \mapsto \mathrm{tor}(M, \varrho)$$
$$\mathrm{Vol}\colon [\varrho] \mapsto \mathrm{Vol}(\varrho)$$

with $\mathrm{Vol}(\rho) = \mathrm{Vol}(M)$ when ρ is the holonomy.

Deformation

A natural question is then:

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Can we "deform" the statement of Müller and Menal-Ferrer–Porti into:

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for any $\varrho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ *close to the holonomy representation* ρ*?*

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It turns out that most of the techniques of their proofs fall down when ϱ is not the holonomy representation.

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$$\begin{aligned} \rho_n \otimes \chi_\zeta \colon \pi_1(M) &\to \mathrm{SL}_n(\mathbb{C}) \otimes \mathbb{S}^1 \\ \gamma &\mapsto \rho_n(\gamma)\chi_\zeta(\gamma) \end{aligned}$$

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We will consider the torsions of the twisted representations $\mathrm{tor}(M, \rho_n \otimes \zeta)$.

Twisted Alexander polynomials

There is a family of polynomials $\Delta^n_M(t) \in \mathbb{C}[t^{\pm 1}]$, the ρ_n-twisted Alexander polynomials, that refine the construction of the Alexander polynomial for knots (Wada, Lin...)

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Our first result is the following:

Theorem (BDHP '19)

For any ζ on the unit circle $\mathbb{S}^1$,

$$|\Delta_M^n(\zeta)| = |\operatorname{tor}(M, \rho_n \otimes \chi_\zeta)|.$$

In particular, the polynomials Δ_M^n have no roots on the unit circle.

A suggestive computation

Those polynomials satisfy a lot of nice symmetry properties. Among them, if $\overline{M} \to M$ is a cyclic k-sheeted covering map then for any t

$$\Delta^n_{\overline{M}}(t) = \prod_{\zeta^k=1} \Delta^n_M(\zeta t)$$

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Taking the limit as $n \to \infty$ and applying previous theorem:

$$\frac{2\,\mathrm{Vol}(M)}{4\pi} = \frac{\mathrm{Vol}(M)}{4\pi} + \lim_{n\to\infty} \frac{\log|\Delta^n_M(-1)|}{n^2}$$

Our main theorem

One deduces

$$\lim_{n\to\infty} \frac{\log|\Delta_M^n(\zeta)|}{n^2} = \frac{\mathrm{Vol}(M)}{4\pi}$$

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Theorem (BDHP '19)

For any ζ on the unit circle,

$$\lim_{n\to\infty} \frac{\log|\Delta_M^n(\zeta)|}{n^2} = \frac{\mathrm{Vol}(M)}{4\pi}$$

uniformly in ζ.

Cheeger–Müller theorem

As we said before, to prove this theorem we need to study the asymptotic of the Reidemeister torsions $\mathrm{tor}(M, \rho_n \otimes \chi_\zeta)$ as n goes to ∞.

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Theorem (Cheeger '77, Müller '78, '91, Bismut–Zhang '91)

Let N be a compact (3-)manifold, $\varrho \colon \pi_1(N) \to \mathrm{GL}_n(\mathbb{C})$ a unimodular representation (i. e. $|\det \rho(\gamma)| = 1$ for any γ in $\pi_1(N)$).
Let $T(M, E_\varrho)$ denote the analytic torsion of the flat bundle E_ϱ associated to ϱ. Then

$$|\mathrm{tor}(M, \varrho)| = T(M, E_\varrho).$$

Idea of the proof: the analytic torsion.

For the proof we assume that M is compact, so that we can use the Cheeger–Müller theorem: since $\rho_n \otimes \chi_\zeta$ is unimodular, we are led to consider the sequence of analytic torsions

$$\left(T(M, E_{\rho_n \otimes \chi_\zeta})\right)_n.$$

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- The flat vector bundle $E_{\rho_n \otimes \chi_\zeta}$ is the quotient $\mathbb{H}^3 \times_{\rho_n \otimes \chi_\zeta} \mathbb{C}^n$ of the trivial bundle of rank n on $\mathbb{H}^3$ by the equivalence relation

$$(\widetilde{x}, v) \sim (\gamma \cdot \widetilde{x}, \rho_n \otimes \chi_\zeta(\gamma) v)$$

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- An $E_{\rho_n \otimes \chi_\zeta}$-valued function (or differential form) $f : M \to E_{\rho_n \otimes \chi_\zeta}$ is a $\pi_1(M)$-equivariant function $f : \mathbb{H}^3 \to \mathbb{C}^n$. We denote by $\Omega^*(M, E_{\rho_n \otimes \chi_\zeta})$ the complex of $E_{\rho_n \otimes \chi_\zeta}$-valued differential forms on M.

Idea of the proof: the analytic torsion 2.

$$\Omega^0(M, E_{\rho_n \otimes \chi_\zeta}) \xrightarrow{d_0} \Omega^1(M, E_{\rho_n \otimes \chi_\zeta}) \xrightarrow{d_1} \Omega^2(M, E_{\rho_n \otimes \chi_\zeta}) \xrightarrow{d_2} \cdots$$

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Once again, we wish to compute the (analytic) torsion as

$$T(M, E_{\rho_n \otimes \chi_\zeta}) = \prod_i \det(d_i)^{(-1)^i}$$

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The Laplacian is the operator
$\Delta_k = d_{k-1}\, d_{k-1}^* + d_k^* d_k : \Omega^k(M, E_{\rho_n \otimes \chi_\zeta}) \to \Omega^k(M, E_{\rho_n \otimes \chi_\zeta})$.

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Definition

The analytic torsion is defined as

$$T(M, E_{\rho_n\otimes\chi_\zeta}) = \prod_{k=0}^{3} (\det \Delta_k)^{(-1)^k \frac{k}{2}}.$$

Determinant and trace

But what is the meaning of $\det \Delta$? The spectrum of the Laplacian consists of eigenvalues $\{0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m \to +\infty\}$. Formally, we write $\log \det \Delta = \operatorname{Tr} \log \Delta = \sum \log \lambda_m$ but the latter makes no sense.

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$$\int_0^\infty e^{-t\lambda} t^{s-1} dt = \lambda^{-s} \Gamma(s)$$

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$$\int_0^\infty e^{-t\lambda} t^{s-1} dt = \lambda^{-s} \Gamma(s)$$

with $\Gamma(s)$ the gamma-function (meromorphic, with simple pole at 0). The candidate for $\log \det \Delta$ is

$$-\frac{d}{ds} \left(\frac{\int_0^\infty \sum_m e^{-t\lambda_m} t^{s-1} dt}{\Gamma(s)} \right) \Bigg|_{s=0}$$

The heat operator

We have just seen that to make sense of the determinant of the Laplace operator $\det \Delta$, one needs to study the sum $\sum_m e^{-t\lambda_m}$, which is the trace of the heat operator $e^{-t\Delta}$ (and is well-defined for any $t > 0$).

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Denoting $(\Delta^n)_n$ for the family of Laplacians acting of forms valued in the family of bundles $\left(E_{\rho_n \otimes \chi_\zeta}\right)_n$, we need to study the asymptotic behavior of the heat traces of this family as n goes to infinity.

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Since the Laplacians operators are equivariant under the action of $\pi_1(M)$, we can decompose the heat traces on translated of a fundamental domain $\mathcal{F} \subset \mathbb{H}^3$ for M:

$$\operatorname{Tr} e^{-t\Delta^n} = \sum_{[\gamma] \in [\pi_1(M)]} \chi_\zeta(\gamma) \int_{\mathcal{F}} h_t^n(\widetilde{x}, \gamma \cdot \widetilde{x})$$

Ruelle zeta functions

After (many) more computations, one obtains the formula:

$$\log T(M, E_{\rho_n \otimes \chi_\zeta}) = \frac{n^2 \operatorname{Vol}(M)}{4\pi} - \sum_{k=1}^{n} \sum_{[\gamma] \neq 1} \log \left| 1 - \chi_\zeta(\gamma) e^{-\frac{k\lambda(\gamma)}{2}} \right| \tag{1}$$

where $\lambda(\gamma)$ is the complex length of γ

i.e. $\rho(\gamma) \sim \begin{pmatrix} e^{\lambda(\gamma)/2} & 0 \\ 0 & e^{-\lambda(\gamma)/2} \end{pmatrix}$.

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The last series is know as a Ruelle zeta function, and part of the computations goes through a proof of Fried theorem, which relates those Ruelle zeta functions with the analytic torsion. The end of the proof deals with the convergence of the above sum as n goes to ∞. The delicate points are uniformity of the convergence.

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- We also want uniformity in ζ.
- Dividing by n^2 and taking the limit finishes the proof.

A dynamical application of our first theorem

Assume that M is fibered: $M = \Sigma \times [0,1]/(x,0) \sim (\phi(x),1)$ for some surface Σ and some diffeomorphism $\phi \colon \Sigma \to \Sigma$, called the monodromy.

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The representation ρ_n restricts to a representation of $\pi_1(\Sigma)$, and the monodromy acts on $\rho_{n,\Sigma}$ by conjugation.

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We prove the following:

Theorem (BDHP'19)

The action of the monodromy ϕ on $[\rho_{n,\Sigma}]$ has hyperbolic dynamic. Namely its tangent map has no eigenvalues of modulus one.

About the proof

The proof comes from the following well-known fact theorem (due to Weyl): the tangent space of the character variety $X(\Sigma)$ naturally identifies with the first twisted cohomology group

$$T_{[\rho_{n,\Sigma}]}X(\Sigma) \simeq H^1(\Sigma, \mathrm{Ad} \circ \rho_{n,\Sigma})$$

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We conclude with our first theorem.

An application of the main theorem.

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We obtain

Theorem (BDHP'19)

$$\lim_{n\to\infty}\frac{m(\Delta_M^n)}{n^2} = \frac{\mathrm{Vol}(M)}{4\pi}$$

Thank you!