Dynamical zeta functions for geodesic flows and the higher-dimensional Reidemeister torsion for Fuchsian groups

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Purpose

Study the asymptotic behaviour of the R-torsion for

$M = \Gamma \backslash PSL_2(\mathbb{R})$ and $\rho_{2N} = \sigma_{2N} \circ \rho : \pi_1 M \xrightarrow{\rho} SL_2(\mathbb{R}) \xrightarrow{\sigma_{2N}} SL_{2N}(\mathbb{C})$

(a discrete subgr. of finite type)

given by the geometric description

$M = \tilde{\Gamma} \backslash \widetilde{PSL_2}(\mathbb{R})$

$\pi_1 M \cong \tilde{\Gamma} \subset \widetilde{PSL_2}(\mathbb{R}) \subset \mathrm{Isom}\,\widetilde{PSL_2}(\mathbb{R})$

via the Ruelle zeta function $R_{\rho_{2N}}(s)$

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Previous works (& Motivation)

W. Müller (2010)

For a closed hyp. 3-mfd $X = \pi_1 X \backslash \mathbb{H}^3$, it holds that

$$\lim_{n\to\infty} \frac{\log|\mathrm{Tor}(X;\rho_n)|}{n^2} = \frac{\mathrm{Vol}(X)}{4\pi}$$

where $\rho_n : \pi_1 X \overset{\rho}{\hookrightarrow} SL_2(\mathbb{C}) \xrightarrow[\text{symm. prod.}]{(n-1)\text{-th}} SL_n(\mathbb{C})$.

(= n-dim. irred. rep. of $SL_2(\mathbb{C})$)

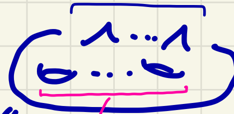
$\boxed{\text{Note}}$

$\rho : \pi_1 X \hookrightarrow SL_2(\mathbb{C})$ is a lift of

the holonomy rep. $\pi_1 X \hookrightarrow PSL_2(\mathbb{C}) \cong \mathrm{Isom}^+ \mathbb{H}^3$

$\left(SL_2(\mathbb{C}) / \{\pm I\} \right)$

Y.

For a closed Seifert 3-mfd M (S^1-b'dle over 

$\| \; \Sigma$)

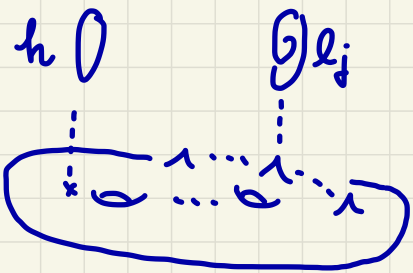
it holds that

$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{2N} = -\left(2-2g-\sum_{j=1}^{m} \frac{\lambda_j-1}{\lambda_j}\right)\log 2$$

$$\leq -\chi^{orb}(\Sigma)\log 2$$

where $\rho_{2N} : \pi_1 M \longrightarrow SL_2(\mathbb{C}) \longrightarrow SL_{2N}(\mathbb{C})$

$h \longmapsto -I$



$\lambda_j = \frac{1}{2}(\text{order of } \rho(\ell_j))$

$\rho(\ell_j) \in SL_2(\mathbb{C})$

- Müller showed that

$$\lim_{n\to\infty} \frac{\log|\mathrm{Tor}(X;\rho_n)|}{n^2} = \frac{\mathrm{Vol}(X)}{4\pi}$$

by using the Ruelle & Selberg zeta fcts.

- Y. showed that

$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{2N} = -\left(2-2g-\sum_{j=1}^{m}\frac{\lambda_j-1}{\lambda_j}\right)\log 2$$

$$\leqq -\chi^{orb}(\Sigma)\log 2$$

by explicit computations on $\mathrm{Tor}(M;\rho_{2N})$.

Main th'm (Y.)

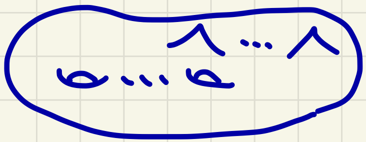
- $|R_{\rho_{2N}}(0)| = \mathrm{Tor}(M; \rho_{2N})$

  (R-torsion for $M = \Gamma \backslash PSL_2(\mathbb{R})$ & ρ_{2N})

- $$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{2N} = \lim_{N\to\infty} \frac{\log|R_{\rho_{2N}}(0)|}{2N}$$

$S^1 \to M = \Gamma \backslash PSL_2(\mathbb{R})$

$\downarrow$

$\Sigma = M/S^1 =$ 

$\Gamma \backslash \mathbb{H}^2$

hyp. orbifold

$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in PSL_2(\mathbb{R}) \right\}$

$$= \underbrace{\frac{\mathrm{Area}(\Sigma)}{2\pi}}_{\text{Gauss-Bonnet}} \log 2$$

$$\left(= - \chi^{orb}(\Sigma) \log 2 \right)$$

In my prev. work, $\displaystyle \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{2N} \leqq -\chi^{orb}(\Sigma)\log 2$

upper bound

◎ On Seifert fibered spaces $\Gamma \backslash PSL_2(\mathbb{R})$

$\Gamma \backslash PSL_2(\mathbb{R}) \cong S^1(T\Sigma)$: the unit tangent b'dle over $\Sigma = \Gamma \backslash \mathbb{H}^2$ 

$$\begin{tikzcd} PSL_2(\mathbb{R}) \arrow[r, phantom, "\cong"] \arrow[d] & S^1(T\mathbb{H}^2) \arrow[d] \\ PSL_2(\mathbb{R}) / S^1 \arrow[r, phantom, "\cong"] & \mathbb{H}^2 \end{tikzcd}$$

$S^1 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$

$\mathbb{H}^2 \circlearrowleft \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \dfrac{a\sqrt{-1}+b}{c\sqrt{-1}+d} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \sqrt{-1}$

the stabilizer of $z = \sqrt{-1}$ $= S^1$

$\left(\{ g \cdot \sqrt{-1} = \sqrt{-1} \mid g \in PSL_2(\mathbb{R}) \} \right)$

$\Gamma \backslash$

$$\begin{array}{ccc} \Gamma \backslash PSL_2(\mathbb{R}) & = & S^1(T\Sigma) \\ \downarrow & & \\ \Gamma \backslash \mathbb{H}^2 & = & \Sigma \end{array}$$

the geometric description of $M = \Gamma \backslash PSL_2(\mathbb{R})$

$$M = \pi_1 M \backslash \widetilde{PSL_2(\mathbb{R})}$$ the univ. cover of $PSL_2(\mathbb{R})$

$\uparrow$ a subgroup of Isom $\widetilde{PSL_2(\mathbb{R})}$

Fact

$$M \cong \tilde{\Gamma} \backslash \widetilde{PSL_2(\mathbb{R})}$$

where $\tilde{\Gamma} \subset \widetilde{PSL_2(\mathbb{R})} \subset \mathrm{Isom}\, \widetilde{PSL_2(\mathbb{R})}$

$\quad \rotatebox{90}{$=$}\ p^{-1}(\Gamma)$, $p : \widetilde{PSL_2(\mathbb{R})} \longrightarrow PSL_2(\mathbb{R})$ proj.

Rem

- $\pi_1 M \cong \tilde{\Gamma}$
- $Z(\tilde{\Gamma}) = Z(\widetilde{PSL_2(\mathbb{R})})$
  the center
  $\cong \mathbb{Z}$

$$1 \to \underbrace{\pi_1 PSL_2(\mathbb{R})}_{\cong\, \mathbb{Z}} \to \widetilde{PSL_2(\mathbb{R})} \to PSL_2(\mathbb{R}) \to 1$$

$$\rotatebox{90}{$\rightsquigarrow$}$$

$$1 \to \mathbb{Z} \to \tilde{\Gamma} \to \Gamma \to 1$$

$$1 \longrightarrow \pi_1 PSL_2(\mathbb{R}) \longrightarrow \widetilde{PSL_2}(\mathbb{R}) \xrightarrow{p} PSL_2(\mathbb{R}) \longrightarrow 1 \qquad \left(\begin{array}{l}\text{central}\\ \text{extension}\end{array}\right)$$

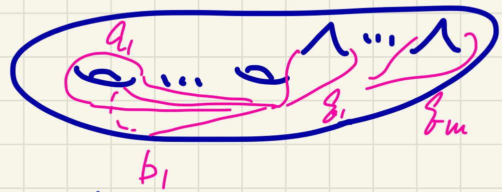
$$\pi_1 PSL_2(\mathbb{R}) = \pi_1(S(T\mathbb{H}^2)) \cong \mathbb{Z} = \langle h \rangle$$

regular fiber

$$\widetilde{PSL_2}(\mathbb{R}) \supset \tilde{\Gamma} = p^{-1}(\Gamma) \longrightarrow \Gamma \subset PSL_2(\mathbb{R})$$

$$\Gamma = \pi_1 \Sigma$$

$$= \langle a_1, b_1, \cdots, a_g, b_g, q_1, \cdots, q_m \mid$$



$$q_j^{\alpha_j} = 1,\ \prod_j q_j \prod_i [a_i, b_i] = 1 \rangle$$

$$\pi_1 M = \tilde{\Gamma}$$

$$= \langle a_1, b_1, \cdots, a_g, b_g, q_1, \cdots, q_m, h \mid$$

h : central,

$$q_j^{\alpha_j} h^{\alpha_j - 1} = 1,$$

$$\prod_j q_j \prod_i [a_i, b_i] = h^{2-2g} \rangle$$

For $$M = \underset{(\pi_1 M)}{\overset{\shortparallel}{\tilde{\Gamma}}} \backslash \widetilde{PSL_2(\mathbb{R})},$$

the restriction of $P' : \widetilde{PSL_2(\mathbb{R})} \longrightarrow SL_2(\mathbb{R})$ to $\tilde{\Gamma}$

gives

$$\rho = P'|_{\tilde{\Gamma}} : \tilde{\Gamma}\ (= \pi_1 M) \longrightarrow SL_2(\mathbb{R})$$

Lem

$$\rho(\underset{Z(\widetilde{PSL_2(\mathbb{R})})}{\overset{\shortparallel}{Z(\tilde{\Gamma})}}) = \{\pm I\} = Z(SL_2(\mathbb{R})) \quad (= Z(SL_2(\mathbb{C})))$$

$\rightsquigarrow$ the R-torsion $\mathrm{Tor}(M; \underset{\sigma_{2N} \circ \rho \,:\, \pi_1 M \to SL_2(\mathbb{R}) \to SL_{2N}(\mathbb{C})}{\overset{\shortparallel}{\underline{\rho_{2N}}}})$ is defined for ${}^{\forall}N$.

Ruelle zeta functions for geodesic flows

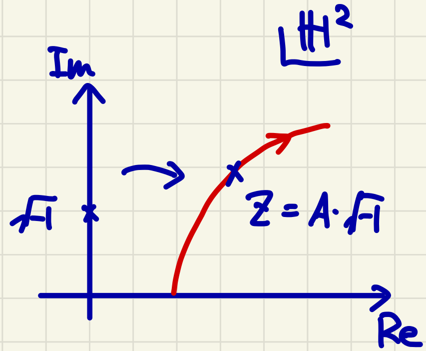
For a Riem. mfd (X, g),

<u>the geodesic flow</u> is the following $\mathbb{R}$-action :

$$\phi_t : S(TX) \times \mathbb{R} \longrightarrow S(TX)$$
$$((x, v), t) \longmapsto (\exp_x(tv), d\exp_x(tv))$$

Fact

$$PSL_2(\mathbb{R}) \times \mathbb{R} \longrightarrow PSL_2(\mathbb{R})$$
$$(A, t) \longmapsto A \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$



$\rightsquigarrow$ $\Gamma \backslash$ $\quad \Gamma \backslash PSL_2(\mathbb{R}) \times \mathbb{R} \longrightarrow \Gamma \backslash PSL_2(\mathbb{R})$

also gives the geodesic flow on $\Gamma \backslash PSL_2(\mathbb{R})$

Def

- the **Ruelle zeta function** for the geodesic flow

$$R_{\rho_{2N}}(s) = \prod_{\substack{\gamma:\text{ prime} \\ \text{closed orbit of } \phi_t}} \det\left(I_{2N} - \rho_{2N}(\gamma) e^{-s\,\underline{l(\gamma)}} \right)$$

$l(\gamma)$ = the period of γ (= the length of γ)

- the **Selberg zeta function** for Γ

$$Z(s) = \prod_{\{T\}} \prod_{k=0}^{\infty} \left(1 - N(T)^{-s-k} \right) \quad (\mathrm{Re}(s) > 1)$$

$\{T\}$: the conj. class of a prime $\underline{\text{hyp. } T \neq 1}$ in Γ

$\underline{\text{hyp. } T} \sim \begin{pmatrix} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{pmatrix}$ $\quad N(T) = e^{\lambda} \ (>1)$

Relation between the Ruelle and Selberg zeta fct

$$R_{\rho_{2\nu}}(s) = \prod_{\substack{\gamma:\ \text{prime} \\ \text{closed orbit of } \phi_t}} \det\Big(I_{2\nu} - \underbrace{\rho_{2\nu}(\gamma)}_{\text{conj.}} e^{-s\underbrace{l(\gamma)}_{\text{the length of } \gamma}} \Big)$$

$$\rho(\gamma) \underset{\text{conj.}}{\sim} \begin{pmatrix} e^{\frac{l(\gamma)}{2}} & 0 \\ 0 & e^{-\frac{l(\gamma)}{2}} \end{pmatrix}$$

hyp. elem

$$\begin{pmatrix} e^{\frac{l(\gamma)}{2}(-2\nu+1)} & & & & & \\ & \ddots & & & & \\ & & e^{-\frac{l(\gamma)}{2}} & & & \\ & & & e^{\frac{l(\gamma)}{2}} & & \\ & & & & \ddots & \\ & & & & & e^{\frac{l(\gamma)}{2}(2\nu-1)} \end{pmatrix}$$

$$= \prod_{k=1}^{\nu} \prod_{\gamma} \Big(1 - e^{-(s+\frac{2k-1}{2})l(\gamma)}\Big) = \boxed{\frac{Z_1(s+\frac{2k-1}{2})}{Z_1(s+\frac{2k-1}{2}+1)}}$$

$$\times \prod_{k=1}^{\nu} \prod_{\gamma} \Big(1 - e^{-(s-\frac{2k-1}{2})l(\gamma)}\Big)$$

$$\| \quad \boxed{\frac{Z_1(s-\frac{2k-1}{2})}{Z_1(s-\frac{2k-1}{2}+1)}}$$

remain

$$= \frac{\cancel{Z_1(s+\frac{1}{2})}}{\cancel{Z_1(s+\frac{3}{2})}} \cdot \frac{\cancel{Z_1(s+\frac{3}{2})}}{Z_1(s+\frac{5}{2})} \cdots \boxed{///} \cdot \frac{\cancel{Z_1(s-\frac{1}{2})}}{\cancel{Z_1(s+\frac{1}{2})}} \; \frac{Z_1(s-\frac{3}{2})}{\cancel{Z_1(s-\frac{1}{2})}} \cdots \boxed{///}$$

Prop (Y.)

- $$R_{\rho_{2N}}(s) = \frac{Z_1(s-N+\frac{1}{2})}{Z_1(s+N+\frac{1}{2})}$$

- $$R_{\rho_{2N}}(0)^{-1} = \frac{Z_1(N+\frac{1}{2})}{Z_1(1-(N+\frac{1}{2}))} \leftarrow \frac{Z_1(s)}{Z_1(1-s)} \text{ at } s=N+\tfrac{1}{2}$$

Functional eq. of $Z_1(s)$

fct defined by the identity elem.

$$\frac{d}{ds}\log\frac{Z_1(s)}{Z_1(1-s)} = \mu(\mathcal{D})\,(s-\tfrac{1}{2})\tan\pi(s-\tfrac{1}{2})$$

$$+\sum_{\{R\}} \frac{-\pi}{m(R)\sin\theta(R)}\,\frac{\cos(2\theta(R)(s-\frac{1}{2}))}{\cos\pi(s-\frac{1}{2})}$$

fct. defined by hyp. elem. in Γ

$R \sim \begin{pmatrix}\cos\theta(R) & -\sin\theta(R)\\ \sin\theta(R) & \cos\theta(R)\end{pmatrix}$ of order $m(R)$ in Γ

$$R_{\rho_{2r}}(0)^{-1} = \exp \int_{1/2}^{N+1/2} \frac{d}{ds} \log \frac{Z_1(s)}{Z_1(1-s)}$$

contrib. of id

$$= \overbrace{\exp \int_{1/2}^{N+1/2} \mu(\mathcal{D})\,(s-1/2) \tan \pi (s-1/2)\, ds}^{\text{contrib. of id}}$$

$$\cdot \underbrace{\exp \int_{1/2}^{N+1/2} \sum_{\{R\}} \frac{-\pi}{m(R) \sin \theta(R)} \frac{\cos\left(2\theta(R)(s-1/2)\right)}{\cos \pi (s-1/2)}\, ds}_{\text{contrib. of elliptic elem.s in } \Gamma}$$

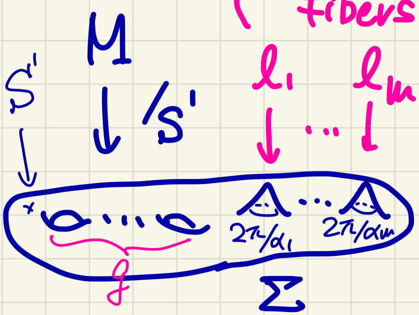
contrib. of elliptic elem.s in Γ

Th'm (Y.)

$$|R_{\rho_{2r}}(0)| = \mathrm{Tor}(M; \rho_{2r})$$

☺ This follows by comparing $|R_{\rho_{2r}}(0)|$ with $\mathrm{Tor}(M; \rho_{2r})$ in my previous result.

$$\mathrm{Tor}(M;\rho_{2N}) = \underline{2^{-2N(2-2g-m)}} \; \underline{\prod_{j=1}^{m}\prod_{k=1}^{N}\left(2\sin\frac{\pi(2k-1)}{2\alpha_j}\right)^{-2}}$$



decompo.

$$\left(\Sigma \setminus \coprod_{j=1}^{m} \mathring{D}^2_j\right) \times \underline{S^1} \cup \underline{N(l_1)} \cup \cdots \cup N(l_m)$$

S^1: regular fiber

$N(l_1) \cong$ Solid torus

$$\mathrm{Tor}(W \times S^1;\rho_{2N}) = \mathrm{Tor}(S^1;\rho_{2N})^{\chi(W)}$$

$$= \det\big(\underbrace{\rho_{2N}(S^1)}_{= -I_{2N}} - I_{2N}\big)^{-\chi(W)}$$

$$= (-2)^{-2N\underbrace{\chi(W)}_{= 2-2g-m}} \qquad W = \Sigma \setminus \coprod_{j=1}^{m} \mathring{D}^2_j$$

$$|R_{\rho_{2N}}(0)|^{-1} = \left| \exp \int_{1/2}^{N+1/2} \frac{d}{ds} \log \frac{Z_1(s)}{Z_1(1-s)} \right|$$

$$= \left| \exp \int_{1/2}^{N+1/2} \mu(\mathcal{D}) \left(s-\tfrac{1}{2}\right) \tan \pi \left(s-\tfrac{1}{2}\right) ds \right|$$

$$\times \left| \exp \int_{1/2}^{N+1/2} \sum_{\{R\}} \frac{-\pi}{m(R) \sin \theta(R)} \frac{\cos(2\theta(R)(s-1/2))}{\cos \pi (s-1/2)} ds \right|$$

$$= \exp\Big(\underbrace{\mu(\mathcal{D})}_{= -2\pi \chi^{orb}(\Sigma)} \Big(-\frac{N}{\pi} \log 2\Big)\Big)$$

$\mu(\mathcal{D})$: the measure of the fundamental domain $\mathcal{D}$ for Σ

∵ Gauss-Bonnet

$\chi^{orb}(\Sigma) = 2-2g - \sum_{j=1}^{m} \frac{\alpha_j - 1}{\alpha_j}$

$$\times \exp\left(\sum_{j=1}^{m} \left(\log \prod_{k=1}^{N} \left(2 \sin \frac{\pi(2k-1)}{2\alpha_j}\right)^2 - \frac{2N}{\alpha_j} \log 2 \right)\right)$$

$$= 2^{2N(2-2g-m)} \prod_{j=1}^{m} \prod_{k=1}^{N} \left(2 \sin \frac{\pi(2k-1)}{2\alpha_j} \right)^2$$

$$= \mathrm{Tor}(M; \rho_{2N})^{-1}$$

//

◎ Relation to the asymptotics of $\mathrm{Tor}(M;\rho_{2N})$

$$\lim_{N\to\infty}\frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{2N} = \lim_{N\to\infty}\frac{\log|R_{\rho_{2N}}(0)|}{2N}$$

$$= \lim_{N\to\infty}\frac{\log|\text{contrib. of the identity}|}{2N}$$

$$+ \underbrace{\lim_{N\to\infty}\frac{\log|\text{contrib. of elliptic elem.}|}{2N}}_{=\,0}$$

$$= \lim_{N\to\infty}\frac{\mu(D)\frac{N}{\pi}\log 2}{2N}$$

the measure of the fundamental domain D for Σ

$$= \underbrace{\frac{\mu(D)}{2\pi}}_{=\,-\chi^{orb}(\Sigma)}\log 2$$

by Gauss-Bonnet

$$\lim_{N\to\infty} \frac{\log |\text{contrib. of the identity}|}{2N}$$

$$\left(\because \; |\text{contrib. of the identity}| = \exp\left(\underline{\mu(D)\frac{N}{\pi}\log 2}\right) \right)$$

We can see that the order is N

$$= \lim_{N\to\infty} \frac{\mu(D)\frac{N}{\pi}\log 2}{2N}$$

$$= \frac{\mu(D)}{2\pi}\log 2$$

$$\lim_{N\to\infty} \frac{\log|\text{contrib. of elliptic elem.}|}{2N}$$

$$\left(\because\ |\text{contrib. of elliptic elem.}| = \exp\left(\sum_{j=1}^{m} \log \prod_{k=1}^{N} \left(2\sin\frac{(2k-1)\pi}{2\alpha_j}\right)^{-2} + \frac{2N}{\alpha_j}\log 2 \right) \right.$$

R-torsion of the except. fiber ℓ_j

$$= \lim_{N\to\infty} \sum_{j=1}^{m} \left(\frac{-1}{N} \log \prod_{k=1}^{N} 2\sin\frac{(2k-1)\pi}{2\alpha_j} + \frac{1}{\alpha_j}\log 2 \right)$$

$$= \sum_{j=1}^{m} \left(\underbrace{-\lim_{N\to\infty} \frac{\log \prod_{k=1}^{N} 2\sin\frac{(2k-1)\pi}{2\alpha_j}}{N}}_{=\frac{1}{\alpha_j}\log 2} + \frac{1}{\alpha_j}\log 2 \right)$$

$$= 0$$