Mapping class groups of surfaces and finite type invariants of 3-manifolds

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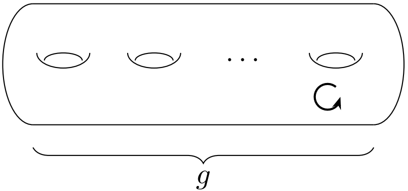
October 30, 2025

based on two joint works
with Y. Nozaki and M. Suzuki
and with Q. Faes and G. Massuyeau

Section 1. The mapping class group and the Torelli subgroup

Notation

$g \geq 1$,

$\Sigma_g :=$  ,

$$\mathcal{M}_g = \{f\colon \Sigma_g \to \Sigma_g : \text{ori. pres. homeomorphism}\}/\text{isotopy}$$
$$= \pi_0 \mathrm{Homeo}_+ \Sigma_g : \text{the mapping class group of } \Sigma_g.$$

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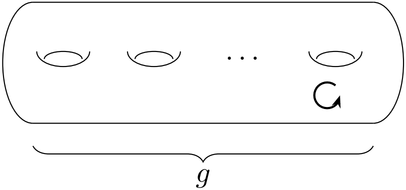
$\Sigma_g :=$ 

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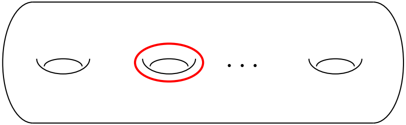
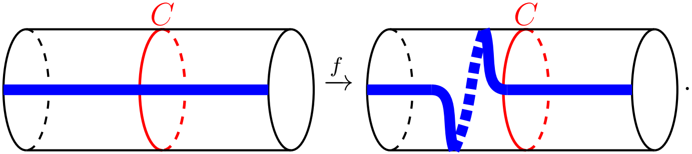
$$\mathcal{M}_g = \{f : \Sigma_g \to \Sigma_g : \text{ori. pres. diffeomorphism}\}/\text{smooth isotopy}$$
$$= \pi_0 \operatorname{Diff}_+ \Sigma_g : \text{the mapping class group of } \Sigma_g.$$

Dehn twists (a generating set of $\mathcal{M}$)

Let C be a simple closed curve in Σ_g. 

Take a tubular neighborhood $N(C)$,

and define $f\colon N(C) \to N(C)$ by twisting once along C:



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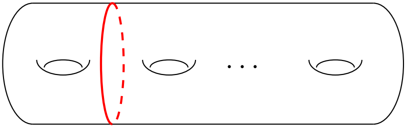

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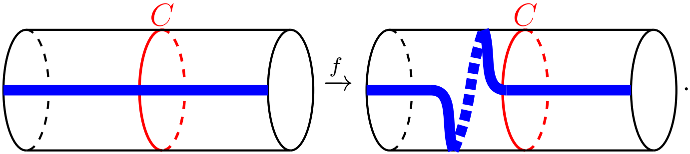

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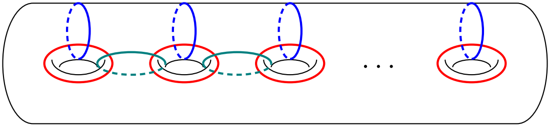
Theorem[Dehn '38, Lickorish'62]

For $g \geq 1$, $\mathcal{M}_g$ is generated by $3g-1$ Dehn twists.



Theorem [Powell '78 (closed surf.), Harer '83 (general), cf. Mumford '67, Birman '70]

For $g \geq 3$, $H_1(\mathcal{M}_g;\mathbb{Z}) = 0$, that is, $\mathcal{M}_g = [\mathcal{M}_g, \mathcal{M}_g]$.

Theorem [McCool '75(existence), Hatcher-Thurston '80, Wajnryb '83]

$\mathcal{M}_g$ admits a finite presentation.

Fact

$\mathcal{M}_g = \pi_1^{\mathrm{orb}}$(moduli of Riemann surfaces of genus g),
$H^*(\mathcal{M}_g;\mathbb{Z}) \cong H^*(\mathrm{BDiff}_+\,\Sigma_g;\mathbb{Z}) \cong H^*(\mathrm{BHomeo}_+\,\Sigma_g;\mathbb{Z})$ $(g \geq 2)$.

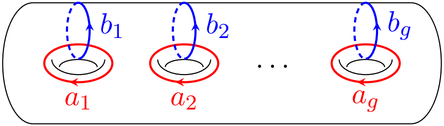
Theorem [Madsen-Weiss '07]

$$\varprojlim_g H^*(\mathcal{M}_g;\mathbb{Q}) \cong \mathbb{Q}[e_1, e_2, \ldots],$$

where $\deg e_i = 2i$.

Symplectic representation

Let a_i, b_i be oriented simple closed curves below.



The intersection form $\mu\colon H_1(\Sigma_g;\mathbb{Z}) \times H_1(\Sigma_g;\mathbb{Z}) \to \mathbb{Z}$ is given by

$$\mu(a_i, a_j) = \mu(b_i, b_j) = 0, \mu(a_i, b_j) = -\mu(b_j, a_i) = \delta_{ij}.$$

The $\mathcal{M}_g$-action on $H_1(\Sigma_g;\mathbb{Z})$ preserves μ, and we have a surjective homomorphism

$$\mathcal{M}_g \to \mathrm{Aut}(H_1(\Sigma_g;\mathbb{Z}), \mu) \cong \mathrm{Sp}(2g, \mathbb{Z}) = \{X \in \mathrm{GL}(2g, \mathbb{Z}) \mid {}^T X J X = J\},$$

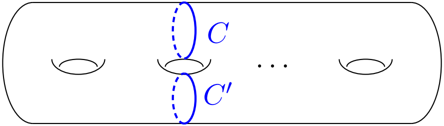
where $J = \begin{pmatrix} O_g & I_g \\ -I_g & O_g \end{pmatrix}$.

The Torelli group $\mathcal{I}_g$

$\mathcal{I}_g := \mathrm{Ker}(\mathcal{M} \to \mathrm{Aut}(H_1(\Sigma_g;\mathbb{Z}),\mu) \cong \mathrm{Sp}(2g;\mathbb{Z}))$: the Torelli group.
Group-theoretically, the Torelli group $\mathcal{I}_g$ is much more complicated than $\mathcal{M}_g$.
D. Johnson became interested in $\mathcal{I}_g$ because every integral homology 3-sphere can be obtained via a Heegaard decomposition whose gluing map lies in $\mathcal{I}_g$.

Theorem [Johnson '83]

Let $g \geq 3$. The Torelli group $\mathcal{I}_g$ is generated by genus 1 bounding pair maps such as the map $t_C t_{C'}^{-1}$. Moreover, $\mathcal{I}_g$ is finitely generated.



Open problem

Is $\mathcal{I}_g$ finitely presentable?

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Is the Torelli groups of non-orientable surfaces finitely generated?

Theorem [Minahan-Putman '25]

For $g \geq 6$, $H^2(\mathcal{I};\mathbb{Q})$ is finitely generated. Moreover, the cup product

$$\Lambda^2(H^1(\mathcal{I};\mathbb{Q})) \to H^2(\mathcal{I};\mathbb{Q})$$

is surjective.

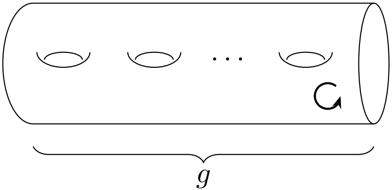
They wrote three papers, which amount to a total of 200 pages.

In the rest of my talk, we will focus on some filtrations of subgroups over $\mathcal{I}$, and their graded quotients can be seen as approximations of $\mathcal{I}$ by Lie algebras.

Section 2. The Johnson filtration and the lower central series of the Torelli group

In the following, we consider a surface with connected boundary

$\Sigma = \Sigma_{g,1} =$



.

$\pi := \pi_1(\Sigma, *)$ is isomorphic to the free group of rank $2g$,

$\mathcal{M} = \mathcal{M}_{g,1} :=$

$\{$ori.-pres. homeo.'s of Σ fixing $\partial\Sigma$ pointwise$\}$/isotopy rel $\partial\Sigma$.

The associated graded of a lower central series

G: a group,

$G = \Gamma_1 G \supset \Gamma_2 G \supset \Gamma_3 G \supset \cdots$: the lower central series defined by $\Gamma_n G = [G, \Gamma_{n-1} G]$.

The associated graded Lie algebra is $\operatorname{Gr}^\Gamma G = \bigoplus_{n=1}^{\infty} \dfrac{\Gamma_n G}{\Gamma_{n+1} G}$.

For $a \in \Gamma_m G/\Gamma_{m+1} G$ and $b \in \Gamma_n G/\Gamma_{n+1} G$, the Lie bracket is defined by group commutators: $[a, b] := aba^{-1}b^{-1} \in \Gamma_{m+n} G/\Gamma_{m+n+1} G$.

Example

1. If G is perfect, $\operatorname{Gr}^\Gamma G = 0$.
2. F_m: the free group of rank m.

   $$\frac{\Gamma_n F_m}{\Gamma_{n+1} F_m} \cong \mathrm{Lie}_n(H_1(F_m; \mathbb{Z})),$$
   $$[\alpha_1, [\alpha_2, \cdots, [\alpha_{n-1}, \alpha_n] \cdots]] \mapsto [\overline{\alpha}_1, [\overline{\alpha}_2, \cdots, [\overline{\alpha}_{n-1}, \overline{\alpha}_n] \cdots]].$$

   Hence, $\operatorname{Gr}^\Gamma F_m = \mathrm{Lie}(H_1(F_m; \mathbb{Z}))$.

Dehn-Nielsen-Baer-Zieschang Theorem

Theorem[Dehn, Nielsen, Baer, Zieschang]

The natural homomorphism

$$\rho\colon \mathcal{M} \to \operatorname{Aut} \pi$$

is injective, and

$$\rho(\mathcal{M}) = \{\varphi \in \operatorname{Aut} \pi \mid \varphi(\zeta) = \zeta\},$$

where $\zeta \in \pi$ is the boundary curve.

In the following, we review the Johnson homomorphism, whch is regarded as a linear approximation of the above map restricted $\mathcal{I}$:

$$\bigoplus_{n=1}^{\infty} \frac{\mathcal{M}(n)}{\mathcal{M}(n+1)} \hookrightarrow \operatorname{Der}_+(\operatorname{Lie}(H)),$$

where $\{\mathcal{M}(n)\}_{n=1}^{\infty}$ is a filtration called the Johnson filtration.

Dennis Johnson's works on the Torelli group

$\pi := \pi_1(\Sigma_{g,1}, *)$, $H := H_1(\Sigma_{g,1}; \mathbb{Z}) = \pi/\Gamma_2\pi$.

Definition (the first Johnson homomorphism)

The map

$$\tau_1 \colon \mathcal{I} \to \mathrm{Hom}\Big(\pi, \frac{\Gamma_2\pi}{\Gamma_3\pi}\Big)$$

defined by $\varphi \mapsto (x \mapsto \varphi(x)x^{-1})$ is a homomorphism.

Theorem [Johnson '80, '85]

- $\mathrm{Im}\,\tau_1 \cong \Lambda^3 H$,
- $\mathrm{Ker}(\tau_1 \colon H_1(\mathcal{I}; \mathbb{Z}) \to H \otimes \Lambda^2 H)$ consists of order 2 elements (detected by the Rochlin invariant of integral homology 3-spheres).

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Let $\mathcal{M}(n) := \operatorname{Ker}\Big(\mathcal{M} \to \operatorname{Aut}\Big(\dfrac{\pi}{\Gamma_{n+1}\pi}\Big)\Big)$. Especially, $\mathcal{M}(1) = \mathcal{I}$.
We also have the n-th Johnson homo.

$$\tau_n \colon \mathcal{M}(n) \to \operatorname{Hom}\Big(\pi, \frac{\Gamma_{n+1}\pi}{\Gamma_{n+2}\pi}\Big), \varphi \mapsto (x \mapsto \varphi(x)x^{-1})$$

which is $\mathcal{M}$-equivariant. By definition, $\mathcal{M}(n)/\mathcal{M}(n+1) \cong \operatorname{Im}\tau_n$.
The Johnson homomorphism is regarded as a linear approximation of the Dehn-Nielsen embedding $\rho \colon \mathcal{M} \to \operatorname{Aut}\pi$, since it induces an embedding

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The Johnson homomorphisms are also written in terms of 3-dimensional invariants:

Remark

- T. Kitano showed that $\tau_n \colon \mathcal{M}(n) \to H \otimes \mathrm{Lie}_{n+1}(H)$ is written by the Massey product on mapping tori.
- A. Heap expressed the Johnson homomorphism in terms of the bordism class in $\Omega_3\left(\dfrac{\pi}{\pi(n+1)}\right)$ represented by mapping tori.
- N. Habegger showed that the Johnson homomorphism can be expressed in terms of the Milnor invariant of string links.

Two filtrations of the Torelli group

$\mathcal{I} = \Gamma_1\mathcal{I} \supset \Gamma_2\mathcal{I} \supset \cdots$: the lower central series,

$\mathcal{I} = \mathcal{M}(1) \supset \mathcal{M}(2) \supset \cdots$: the Johnson filtration,

where $\mathcal{M}(n) = \mathrm{Ker}\Big(\mathcal{M} \to \mathrm{Aut}\Big(\dfrac{\pi}{\Gamma_{n+1}\pi}\Big)\Big)$.

Theorem [Ershov-He '18, Church-Ershov-Putman '21]

Let $n \geq 3$ and $g \geq 2n-1$.

Every subgroup of $\mathcal{I}$ containing $\Gamma_n\mathcal{I}$ is finitely generated.

Fact

1. $\Gamma_n\mathcal{I} \subset \mathcal{M}(n)$,
2. $\mathrm{Sp}(2g,\mathbb{Z}) \cong \mathcal{M}/\mathcal{I}$ acts on $\Gamma_n\mathcal{I}/\Gamma_{n+1}\mathcal{I}$ and $\mathcal{M}(n)/\mathcal{M}(n+1)$.

Main topic of this talk

Problem

1. Determine the $\mathrm{Sp}(2g,\mathbb{Z})$-modules

$$\bigoplus_{n\geq 1}\frac{\Gamma_n\mathcal{I}}{\Gamma_{n+1}\mathcal{I}} \quad \text{and} \quad \bigoplus_{n\geq 1}\frac{\mathcal{M}(n)}{\mathcal{M}(n+1)} \cong \mathrm{Im}\,\tau_n \text{ (torsion-free).}$$

. 2. Determine the kernels and images of the homomorphisms

$$\mathrm{inc}_*\colon \bigoplus_{n\geq 1}\frac{\Gamma_n\mathcal{I}}{\Gamma_{n+1}\mathcal{I}} \to \bigoplus_{n\geq 1}\frac{\mathcal{M}(n)}{\mathcal{M}(n+1)}.$$

Graded quotients tensored with $\mathbb{Q}$

Theorem [Hain '97]

For $g \geq 3$, the natural homomorphism

$$\mathrm{inc}_* \colon \frac{\Gamma_n \mathcal{I}}{\Gamma_{n+1}\mathcal{I}} \otimes \mathbb{Q} \to \frac{\mathcal{M}(n)}{\mathcal{M}(n+1)} \otimes \mathbb{Q}$$

is surjective.

Theorem [Kupers, Randal-Williams '23]

When $g \gg 0$,

$$\mathrm{Ker}\Big(\mathrm{inc}_* \colon \frac{\Gamma_n \mathcal{I}}{\Gamma_{n+1}\mathcal{I}} \otimes \mathbb{Q} \to \frac{\mathcal{M}(n)}{\mathcal{M}(n+1)} \otimes \mathbb{Q}\Big)$$

are trivial $\mathrm{Sp}(2g, \mathbb{Q})$-representations for all n.

n	1	2	3	$\cdots$
$\frac{\Gamma_n \mathcal{I}}{\Gamma_{n+1}\mathcal{I}} \otimes \mathbb{Q}$	$[1^3]+[1]$	$[2^2]+[1^2]+2[0]$	$[31^2]+[21]$	$\cdots$
$\frac{\mathcal{M}(n)}{\mathcal{M}(n+1)} \otimes \mathbb{Q}$	$[1^3]+[1]$	$[2^2]+[1^2]+[0]$	$[31^2]+[21]$	$\cdots$

There is a natural surjective homomorphism

$$J^{\mathbb{Q}}\colon \operatorname{Lie}(\Lambda^3 H_{\mathbb{Q}}) = \operatorname{Lie}(H_1(\mathcal{I};\mathbb{Q})) \to \operatorname{Gr}^{\Gamma}\mathcal{I}\otimes\mathbb{Q},$$

defined by $J^{\mathbb{Q}}([\bar{\varphi}_1,[\cdots,[\bar{\varphi}_{n-1},\bar{\varphi}_n]\cdots]]) = [\varphi_1,[\cdots,[\varphi_{n-1},\varphi_n]\cdots]]$.

Theorem [Hain '97, cf. Kupers, Randal-Williams '23]

The kernel of $J^{\mathbb{Q}}\colon \operatorname{Lie}(\Lambda^3 H_{\mathbb{Q}}) \to \operatorname{Gr}^{\Gamma}\mathcal{I}\otimes\mathbb{Q}$ is generated by some module $R^{\perp}\subset \operatorname{Lie}_2(\Lambda^3 H)$ as a Lie ideal for $g\geq 4$.

Theorem [Garoufalidis-Getzler '17 cf. Kupers, Randal-Williams '23]

There is an $\operatorname{Sp}(2g,\mathbb{Q})$-character formula for $\operatorname{Gr}^{\Gamma}\mathcal{I}_g\otimes\mathbb{Q}$ for the closed surface Σ_g.

In conclusion, rationally, $\mathcal{I}(n)/\mathcal{I}(n+1)$ and $\mathcal{M}(n)/\mathcal{M}(n+1)$ are almost determined.

For small n, the $\mathrm{Sp}(2g, \mathbb{Q})$-module $(\mathcal{M}(n)/\mathcal{M}(n+1)) \otimes \mathbb{Q}$ is computed by Johnson ($n = 1$), Hain, Morita ($n = 2$), Hain, Asada-Nakamura ($n = 3$), Morita ($n = 4$), Morita-Sakasai-Suzuki ($n = 5, 6, 7, 8$).

Corollary

Let $g \gg 0$.

$$\frac{\mathcal{I}(n)}{\mathcal{I}(n+1)} \otimes \mathbb{Q} \to \frac{M(n)}{M(n+1)} \otimes \mathbb{Q}$$

is an isomorphism when $n = 1$, $3 \leq n \leq 8$. When $n = 2$, the kernel is isomorphic to $\mathbb{Q}$ (coming from the Casson invariant).

The Rochlin invariant (Birman-Craggs)

$\varphi \in \mathcal{I}_g$,

V_g: a genus g handlebody embedded in S^3.

$S(\varphi) := (S^3 \setminus \mathrm{Int}\, V_g) \cup_\varphi V_g$: an integral homology 3-sphere.

The Rochlin invariant of $S(\varphi)$ induces a series of homomorphism

$$\mathcal{I}_g \to \mathbb{Z}/2\mathbb{Z},\ \varphi \mapsto \mathrm{Sign}(W(\varphi))/8,$$

where $W(\varphi)$ is a compact oriented spin 4-manifold bounding $S(\varphi)$.

The Casson invariant (Morita)

In the same way, the Casson invariant of $S(\varphi)$ induces a series of homomorphisms

$$\mathcal{I}(2)/\mathcal{I}(3) \to \mathbb{Q},\ \varphi \mapsto \lambda(S(\varphi)).$$

Corollary [Johnson '85]

$$\operatorname{Ker}\Big(\frac{\mathcal{I}}{\Gamma_2\mathcal{I}} \to \frac{\mathcal{M}(1)}{\mathcal{M}(2)}\Big) = \mathbb{Z}_2^{\binom{2g}{2}+\binom{2g}{1}+\binom{2g}{0}}.$$

Moreover, it is described by the Rochlin invariant of integral homology spheres.

Theorem [Faes-Massuyeau-S. '25]

The module $\dfrac{\mathcal{I}(2)}{\mathcal{I}(3)}$ is torsion-free. Moreover,

$$\operatorname{Ker}\Big(\frac{\mathcal{I}(2)}{\mathcal{I}(3)} \to \frac{\mathcal{M}(2)}{\mathcal{M}(3)}\Big) \cong \mathbb{Z},$$

which is described by the Casson invariant of integral homology 3-spheres.

Proof Compute the kernel of the surjective homo.

$$J_2\colon\ \mathrm{Lie}_2(\Lambda^3 H_{\mathbb{Z}}) \to \frac{\mathcal{I}(2)}{\mathcal{I}(3)},\ [\bar\varphi_1, \bar\varphi_2] \mapsto [\varphi_1, \varphi_2].$$

If we compute the kernel of the homomorphism

$$J_3\colon \operatorname{Lie}_3(\Lambda^3 H_{\mathbb{Z}}) \to \frac{\mathcal{I}(3)}{\mathcal{I}(4)},\ [\bar{\varphi}_1, [\bar{\varphi}_2, \bar{\varphi}_3]] \mapsto [\varphi_1, [\varphi_2, \varphi_3]],$$

we can also determine $\dfrac{\mathcal{I}(3)}{\mathcal{I}(4)}$, but it is very hard.
At present, we know only the following using 3-manifold invariants:

Main Theorem [Nozaki-Suzuki-S. '25]

When $g \gg 0$,
the module $\dfrac{\mathcal{I}(2k-1)}{\mathcal{I}(2k)}$ has 2-torsion elements for $k \geq 1$.
Especially, $\mathrm{inc}_* : \dfrac{\mathcal{I}(2k-1)}{\mathcal{I}(2k)} \to \dfrac{\mathcal{M}(2k-1)}{\mathcal{M}(2k)}$ is not injective.

As we saw, we need 3-manifold invariants!

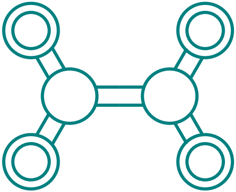
Section 3. Finite-type invariants and $\mathrm{Gr}^{\Gamma}\,\mathcal{I}$

Graph claspers in 3-manifolds

M: compact oriented 3-manifold.

A graph clasper G in M is a (possibly disconnected) surface embedded in M consisting of annuli, rectangles, and disks.

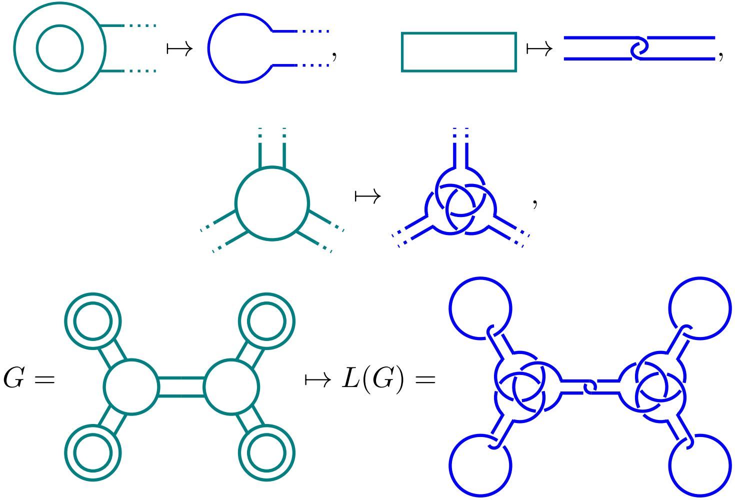
It must have the same shape as a unitrivalent graph, with univalent and trivalent vertices corresponding to annuli and disks, respectively, and edges corresponding to rectangles.

$G =$ 

$\deg G :=$ the number of disks.

Clasper surgery

From a graph clasper G embedded in M, we obtain a framed link $L(G)$.



The surgery along $L(G)$ in M is called the clasper surgery.

Finite type invariants of integral homology 3-spheres

$\mathcal{S} = \{$homeo. classes of integral homology 3-spheres$\}$,

$M \in \mathcal{S}$,

$\Gamma \subset M$: (possibly disconnected) graph clasper. Denote

$$[M, \Gamma] := \sum_{\Gamma' \subset \Gamma} (-1)^{|\Gamma'|} M_{\Gamma'} \in \mathbb{Z}\mathcal{S},$$

where $|\Gamma'| = \#\{$connected components of $\Gamma'\}$. For example,

- if Γ is connected, $[M, \Gamma] = M - M_\Gamma$,
- if $\Gamma = A \amalg B$, $[M, \Gamma] = M - M_A - M_B + M_\Gamma$.

Define

$$\mathcal{F}_n := \langle [M, \Gamma] \mid M \in \mathcal{S}, \Gamma \subset M : \text{a graph clasper}, \deg(\Gamma) = n \rangle .$$

Definition [Finite-type invariants]

A map $v \colon \mathbb{Z}\mathcal{S} \to \mathbb{Q}$ is called a finite-type invariant of degree at most n if $v(\mathcal{F}_{n+1}) = 0$.

Theorem [Garoufalidis-Levine '97, Habiro '00]

For $\varphi \in \mathcal{I}(n)$, $S(\varphi) - S^3 \in \mathcal{F}_n$. Especially, a finite type invariant $v\colon \mathcal{M} \to \mathbb{Q}$ of degree k induces a homomorphism

$$\frac{\mathcal{I}(n)}{\mathcal{I}(n+1)} \to \mathbb{Q},\ \varphi \mapsto v(S(\varphi)) - v(S^3).$$

Corollary (Recall)

Let $g \gg 0$.
$$\frac{\mathcal{I}(n)}{\mathcal{I}(n+1)} \otimes \mathbb{Q} \to \frac{M(n)}{M(n+1)} \otimes \mathbb{Q} \cong \operatorname{Im}\tau_n \otimes \mathbb{Q}$$

is an isomorphism when $n = 1$, $3 \leq n \leq 8$. When $n = 2$, the kernel is isomorphic to $\mathbb{Q}$ (coming from the Casson invariant).

Corollary

Any $\mathbb{Q}$-valued finite-type invariant of degree ≤ 8 does not give a new homomorphism except the Casson invariant. In other words, they are written in terms of Johnson homomorphisms.

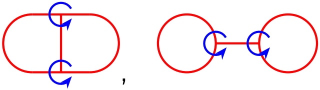
Open problem

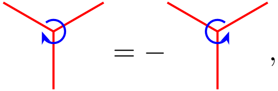
Express the finite-type invariants in terms of the Johnson homomorphisms.

Definition (The space $\mathcal{A}_n(\varnothing)$ (target of the LMO invariant))

$$\mathcal{A}_n(\varnothing) = \frac{\mathbb{Q}\{\text{trivalent graphs with } n \text{ oriented vertices}\}}{(\text{AS, IHX})},$$

where an orientation of a vertex is a cyclic ordering of the three incident edges.


$\in \mathcal{A}_2(\varnothing)$.

$AS:$


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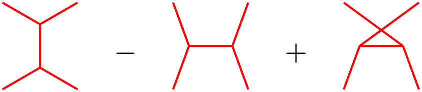
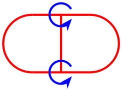
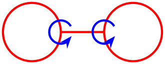

$= 0.$

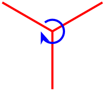
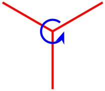
We denote $\widehat{\mathcal{A}}(\varnothing) = \prod_{n=1}^{\infty} \mathcal{A}_n(\varnothing)$.

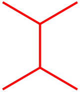
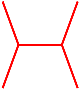
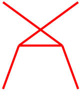
Definition (The space $\mathcal{A}_n(\varnothing)$ (target of the LMO invariant))

$$\mathcal{A}_n^c(\varnothing) = \frac{\mathbb{Q}\{\text{connected trivalent graphs with } n \text{ oriented vertices}\}}{(\text{AS, IHX})},$$

where an orientation of a vertex is a cyclic ordering of the three incident edges.

 ,  $\in \mathcal{A}_2(\varnothing)$.

AS :  $= -$  ,

IHX :  $-$  $+$  $= 0.$

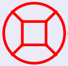
We denote $\widehat{\mathcal{A}}^c(\varnothing) = \prod_{n=1}^{\infty} \mathcal{A}_n^c(\varnothing)$.

The LMO invariant of 3-manifolds

M: connected closed oriented 3-manifold,
$L \subset S^3$: a framed link such that $M \cong (S^3)_L$.
The Kontsevich invariant $Z^K(L)$ lies in $\widehat{\mathcal{A}}(\amalg_{i=1}^{|L|} S^1)$,
where $\widehat{\mathcal{A}}(\amalg_{i=1}^{|L|} S^1)$ is the space of uni-trivalent graphs whose univalent vertices are attached to $\amalg_{k=1}^{|L|} S^1$.
Using the detaching map $\chi^{-1}\colon \widehat{\mathcal{A}}(\amalg_{i=1}^{|L|} S^1) \to \widehat{\mathcal{A}}(\{1, 2, \dots, |L|\})$ etc.,
we obtain $Z^{\mathsf{LMO}}(M) \in \widehat{\mathcal{A}}(\varnothing)$ that is invariant under Kirby moves.

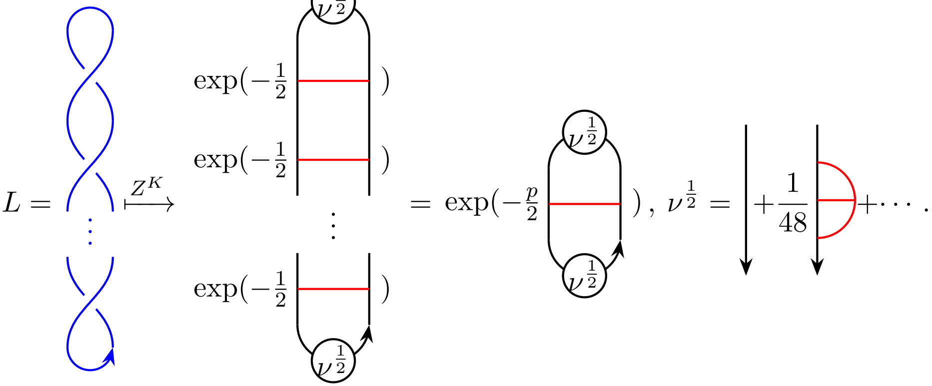
Fact

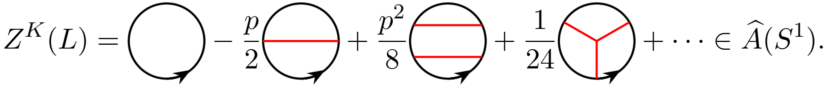
The images of $Z^{\mathsf{LMO}}\colon \{\text{3-mfds}\} \to \widehat{\mathcal{A}}(\varnothing)$ are group-like elements in $\widehat{\mathcal{A}}(\varnothing)$.
In other words, we can express

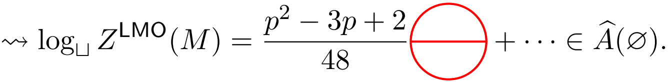
$\log_{\sqcup} Z^{\mathsf{LMO}}(M) = c_1$  $+ c_2$  $+ c_3$  $+ c_{41}$  $+ c_{42}$  $+ \cdots$

where $\exp_{\sqcup} x = \varnothing + x + \frac{1}{2} x \sqcup x + \frac{1}{6} x \sqcup x \sqcup x + \cdots$.

$M = L(p,1) = (S^3)_L$.







(some procedure to make it invariant under the Kirby moves.)

When $M = (S^3)_L$ is a rational homology 3-sphere, it is written as

$$Z^{\mathrm{LMO}}(M) = \mathring{\mathrm{A}}(L) := \mathring{\mathrm{A}}_0(U^+)^{-\sigma_+} \amalg \mathring{\mathrm{A}}_0(U^-)^{-\sigma_-} \amalg \mathring{\mathrm{A}}_0(L),$$

where $U^{\pm}$ is the trivial knot with ± 1 framing,
and $\sigma_{\pm}$ is the number of positive (negative) eigenvalues of $\mathrm{Lk}(L)$.

$\mathring{\mathrm{A}}_0(L)$ is defined by

$$\mathring{\mathrm{A}}_0(L) = \int^{\mathsf{FG}} \chi^{-1} Z^K(L^\nu),$$

where $\chi^{-1} \colon \widehat{\mathcal{A}}(\amalg_{i=1}^{|L|} S^1) \to \widehat{\mathcal{A}}(\{1, 2, \ldots, |L|\})$ is the detaching map,
and $\int^{\mathsf{FG}}$ is some operation to connect strut graphs and the other graphs.

Theorem [Bar-Natan, Garoufalidis, Rozansky, Thurston '02]

The map $\mathring{\mathrm{A}}$ is invariant under the Kirby moves, and hence, it is an invariant of rational homology 3-spheres.

Theorem [Le '97, Habiro '00, Garoufalidis '02] (Universality)

The LMO invariant is universal among the rational valued finite-type invariants of rational homology 3-spheres.
Namely, if a map $v\colon \mathcal{S}^{\mathsf{rat}} \to \mathbb{Q}$ is of finite-type, then there exists $w\colon \widehat{\mathcal{A}}(\varnothing) \to \mathbb{Q}$ such that $v = w \circ Z^{\mathsf{LMO}}$.

Open problem

Are there finite-type invariants of integral homology 3-spheres that provide new information about $\mathcal{I}(n)/\mathcal{I}(n+1)$?
In other words, besides the Casson invariant, are there finite-type invariants that cannot be expressed in terms of the Johnson homomorphisms?

Cheptea-Habiro-Massuyeau extended the LMO invariant to a functor

$$\tilde{Z}\colon \text{(cat. of some 3-dim. cobordisms)} \to \text{(cat. of some Jacobi diagrams)}.$$

The functor gives finite-type invariants of homology cylinders of surfaces.
In the next talk, we will embed the Torelli group to the monoid of homology cylinders $\mathcal{IC}$, and see that the functor gives new homomorphisms on $\mathrm{Gr}^{\Gamma}\,\mathcal{I}$.

$$\begin{array}{ccc} & \mathcal{S} & \xrightarrow[\log_{\sqcup} Z^{\mathsf{LMO}}]{} & \widehat{A}^c(\varnothing) \\ \nearrow & & & \\ \mathcal{I} & & & \\ \searrow & & & \\ & \mathcal{IC} & \xrightarrow{\log_{\sqcup} \tilde{Z}^Y} & \widehat{A}^c(\{1^{\pm},\dots,g^{\pm}\}). \end{array}$$