Associated graded modules of filtrations on the monoid of homology cylinders and the Torelli group

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October 31, 2025

based on two joint works
with Y. Nozaki and M. Suzuki
and with Q. Faes and G. Massuyeau

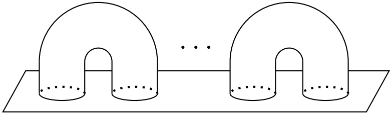
Section 1. An extension of the Torelli group as a monoid

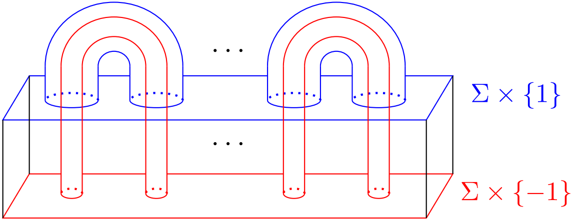
the monoid of homology cylinders $\mathcal{IC}$
$=$ an extension of the Torelli group $\mathcal{I}$,

LMO functor $\tilde{Z}^Y$
$=$ a functorial extension of the LMO invariant Z^{LMO}.

$$\begin{array}{ccc} & \mathcal{S} & \xrightarrow[\log_{\sqcup} Z^{\mathsf{LMO}}]{} & \widehat{A}^c(\varnothing) \\ \nearrow & & & \\ \mathcal{I} & & & \\ \searrow & & & \\ & \mathcal{IC} & \xrightarrow{\log_{\sqcup} \tilde{Z}^Y} & \widehat{A}^c(\{1^{\pm}, \ldots, g^{\pm}\}). \end{array}$$

Homology cylinders

$\Sigma :=$  ,

$\Sigma \times [-1,1] =$ 

.

M: a connected compact oriented 3-manifold s.t. $\partial M \cong \partial(\Sigma \times [-1,1])$,
$m\colon \partial(\Sigma \times [-1,1]) \xrightarrow{\cong} \partial M$: orientation-preserving homeo. satisfying
$(m|_{\Sigma\times\{1\}})_* = (m|_{\Sigma\times\{-1\}})_*\colon H_*(\Sigma;\mathbb{Z}) \xrightarrow{\cong} H_*(M;\mathbb{Z})$.

The pair (M, m) is called a homology cylinder of Σ.

The monoid of homology cylinders

$\mathcal{IC} = \{\text{isom. classes of } (M, m)\}$: the monoid of homology cylinders.

Definition (Isomorphisms between homology cylinders)

(M, m), (N, n): homology cylinders.

If there is a homeomorphism $\Phi\colon M \to N$ such that

$$\begin{array}{ccc} & \partial(\Sigma \times [-1, 1]) & \\ {}^{m}\swarrow & \circlearrowleft & \searrow^{n} \\ \partial M & \xrightarrow[\Phi]{} & \partial N \end{array}$$

we say that (M, m) and (N, n) are isomorphic.

$$(M, m) \circ (N, n) := \begin{array}{|c|}\hline N \\ \hline M \\ \hline\end{array} = M \cup_{m_+(x) = n_-(x)} N, \text{ where}$$

$$m_+ = m|_{\Sigma \times \{1\}} : \Sigma \to \partial M, \; n_- = n|_{\Sigma \times \{-1\}} : \Sigma \to \partial N.$$

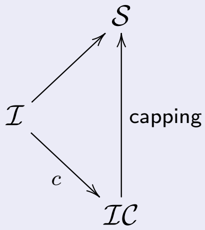
Inclusion $\mathfrak{c}\colon \mathcal{I} \to \mathcal{IC}$

The natural map $\mathfrak{c}\colon \mathcal{I} \to \mathcal{IC}$, $\mathfrak{c}([f]) = (\Sigma \times [-1,1], \mathrm{id}_{\text{bottom and lateral}} \cup f)$ is injective. Thus,

$\mathcal{IC}$ is an extension of $\mathcal{I}$ as a monoid.

Remark

Capping a homology cylinder by two handlebodies of genus g along the top and bottom sides, we obtain a homology 3-sphere.



Definition (Y_n-equivalence)

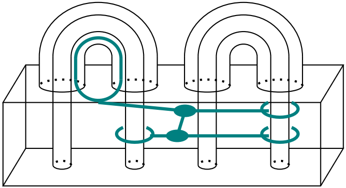
$n \geq 1$,

M, N: connected compact oriented 3-manifolds.

$M \sim_{Y_n} N$ if and only if

N is obtained by surgeries along connected graph claspers $\{G_i\}_{i=1}^{p}$ in M each of which has $\deg = n$, i.e., $N \cong M_{\amalg_{i=1}^{p} G_i}$.

Example

$\Sigma \times [-1, 1] \sim_{Y_2}$  .

The submonoid $Y_n\mathcal{IC}$

$Y_n\mathcal{IC} = \{(M, m) \in \mathcal{IC} \mid (M, m) \sim_{Y_n} (\Sigma \times [-1, 1], \mathrm{id})\}$.

$\mathcal{IC} = Y_1\mathcal{IC} \supset Y_2\mathcal{IC} \supset \cdots$: the Y-filtration.

Theorem [Goussarov '94, Habiro '00]

1. $Y_n\mathcal{IC}/Y_{n+1}$ is a finitely generated abelian group with respect to the composition.
2. $Y_n\mathcal{IC}/Y_{n+p}$ is a finitely generated group for $p \geq 1$.
3. $\displaystyle\bigoplus_{n=1}^{\infty} \frac{Y_n\mathcal{IC}}{Y_{n+1}}$ is a Lie algebra.

   For $M \in Y_m\mathcal{IC}/Y_{m+1}$ and $N \in Y_n\mathcal{IC}/Y_{n+1}$, the Lie bracket is defined by

   $$[M, N] := M \circ N \circ \overline{M} \circ \overline{N} \in Y_{m+n}\mathcal{IC}/Y_{m+n+1},$$

   where $\overline{M} \in Y_m\mathcal{IC}$ represents the inverse element of M in $Y_m\mathcal{IC}/Y_{m+1}$.

The mapping class group $\mathcal{M}$ acts on $Y_n\mathcal{IC}/Y_{n+1}$ by changing the markings from $m_\pm \colon \Sigma \to \partial M$ to $m_\pm \circ \varphi^{-1}$, and it induces an $\mathrm{Sp}(2g,\mathbb{Z})$-action.

Problem

1. Determine the three $\mathrm{Sp}(2g,\mathbb{Z})$-modules

$$\bigoplus_{k\geq 1} \frac{\Gamma_k\mathcal{I}}{\Gamma_{k+1}\mathcal{I}}, \ \bigoplus_{k\geq 1} \frac{\mathcal{M}(k)}{\mathcal{M}(k+1)}, \ \bigoplus_{k\geq 1} \frac{Y_k\mathcal{IC}}{Y_{k+1}}.$$

2. Determine the kernels and images of the homomorphisms

$$\mathrm{inc}_* \colon \bigoplus_{k\geq 1} \frac{\Gamma_k\mathcal{I}}{\Gamma_{k+1}\mathcal{I}} \to \bigoplus_{k\geq 1} \frac{\mathcal{M}(k)}{\mathcal{M}(k+1)},$$

$$\mathfrak{c}_* \colon \bigoplus_{k\geq 1} \frac{\Gamma_k\mathcal{I}}{\Gamma_{k+1}\mathcal{I}} \to \bigoplus_{k\geq 1} \frac{Y_k\mathcal{IC}}{Y_{k+1}}.$$

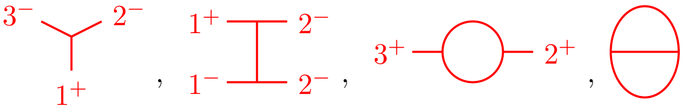
The module $\mathcal{A}_n^c$ of Jacobi diagrams

Definition(Jacobi diagrams)

A Jacobi diagram colored by $\{1^\pm, \ldots, g^\pm\}$ is a uni-trivalent graph s.t.

- each trivalent vertex has a cyclic order of incident edges
- each univalent vertex has a label $\{1^\pm, \ldots, g^\pm\}$.

Example



$$\mathcal{A} = \mathcal{A}(\{1^\pm, \ldots, g^\pm\}) := \frac{\mathbb{Z}\{\text{Jacobi diagrams colored by } \{1^\pm, \ldots, g^\pm\}\}}{(\text{AS, IHX, self-loop})}.$$

$$\deg J = \#\{\text{trivalent vertices in } J\}.$$

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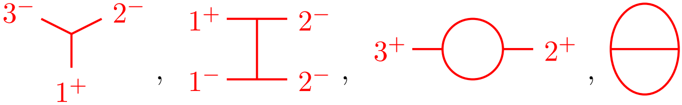


$$\mathcal{A}^c = \mathcal{A}^c(\{1^\pm, \dots, g^\pm\}) := \frac{\mathbb{Z}\{\text{connected Jacobi diagrams colored by } \{1^\pm, \dots, g^\pm\}\}}{(\text{AS, IHX, self-loop})}.$$

$$\deg J = \#\{\text{trivalent vertices in } J\}.$$

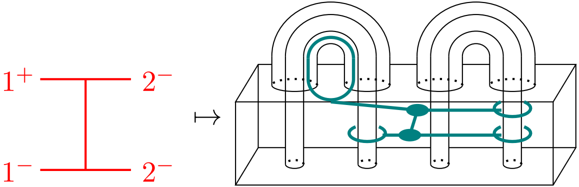
The surgery map

$\mathcal{A}_n^c$: the degree n part of $\mathcal{A}^c$.

Goussarov and Habiro constructed a homomorphism

$$\mathfrak{s}_n \colon \mathcal{A}_n^c \to Y_n\mathcal{IC}/Y_{n+1}$$

called the surgery map.



Fact

1. $\mathfrak{s}_n$ is a module homomorphism,
2. $\mathfrak{s}_n$ is surjective for $n \geq 2$,
3. $\mathfrak{s}_n \otimes \mathrm{id}_{\mathbb{Q}} \colon \mathcal{A}_n^c \otimes \mathbb{Q} \to Y_n\mathcal{IC}/Y_{n+1} \otimes \mathbb{Q}$ is an isomorphism.

Thanks to $\mathfrak{s}_n \colon \mathcal{A}_n^c \twoheadrightarrow Y_n\mathcal{IC}/Y_{n+1}$, the module structure of $Y_n\mathcal{IC}/Y_{n+1}$ is much easier than $\mathcal{I}(n)/\mathcal{I}(n+1)$.

Theorem [Cheptea-Habiro-Massuyeau '08]

For $n \geq 1$,

$$\mathfrak{s}_n \otimes \mathrm{id}_{\mathbb{Q}} \colon \mathcal{A}_n^c \otimes \mathbb{Q} \cong Y_n\mathcal{IC}/Y_{n+1} \otimes \mathbb{Q}.$$

Theorem [Massuyeau-Meilhan '03, '13]

For $g \geq 3$, $\mathcal{IC}/Y_2 \cong \mathcal{I}/\Gamma_2\mathcal{I}\,(\cong \Lambda^3 H_{\mathbb{Z}} \oplus \mathbb{Z}_2^{\binom{2g}{2}+\binom{2g}{1}+\binom{2g}{0}})$, and for $g \geq 1$, $Y_2\mathcal{IC}/Y_3 \cong \mathcal{A}_2^c$ (torsion-free).

Theorem [Faes-Massuyeau-S. '25]

For $g \geq 3$, there is an exact sequence of $\mathrm{Sp}(2g,\mathbb{Z})$-modules

$$0 \longrightarrow \Gamma_2\mathcal{I}/\Gamma_3\mathcal{I} \xrightarrow{\mathfrak{c}} Y_2\mathcal{IC}/Y_3 \xrightarrow{\alpha} S^2H \xrightarrow{C} \mathbb{Z}_2 \longrightarrow 0,$$

where $C(xy) = \mu(x,y) \in \mathbb{Z}_2$.

I expect that $Y_n\mathcal{IC}/Y_{n+1}$ has all information of $\Gamma_n\mathcal{I}/\Gamma_{n+1}\mathcal{I}$.

By investigating the LMO functor further, we computed the torsion parts of $Y_n\mathcal{IC}/Y_{n+1}$.

Theorem [Nozaki-Suzuki-S. '22, '25]

For $g \geq 1$, as abelian groups,

$$\begin{aligned}
\mathrm{tor}(Y_3\mathcal{IC}/Y_4) &\cong \mathrm{Lie}_3(H_{\mathbb{Z}_2}) \oplus S^2 H_{\mathbb{Z}_2}, \\
\mathrm{tor}(Y_4\mathcal{IC}/Y_5) &= 0, \\
\mathrm{tor}(Y_5\mathcal{IC}/Y_6) &= \mathbb{Z}_2^r, \\
\mathrm{tor}(Y_6\mathcal{IC}/Y_7) &\cong \mathbb{Z}_3^s,
\end{aligned}$$

where $4g^3 + 6g^2 \leq r \leq 4g\binom{2g+1}{3} + 4g^3 + 6g^2$ and $\binom{2g+1}{2} \leq s \leq 4g^2$.

Lately, we are also studying the $\mathrm{Sp}(2g,\mathbb{Z})$-module structure.
Let $\tilde{H}_{\mathbb{Z}_2} = H_1(U\Sigma;\mathbb{Z}_2)$ and $\varpi\colon \tilde{H}_{\mathbb{Z}_2} \to H_{\mathbb{Z}_2}$ be the induced homomorphism by the projection $U\Sigma \to \Sigma$.

Theorem [Faes-Massuyeau-S.]

For $g \geq 1$, as $\mathrm{Sp}(2g,\mathbb{Z})$-modules,

$$\mathrm{tor}(Y_3\mathcal{IC}/Y_4) \cong \frac{H_{\mathbb{Z}_2} \otimes \Lambda^2 \tilde{H}_{\mathbb{Z}_2}}{\iota(\Lambda^3 \tilde{H}_{\mathbb{Z}_2})},$$

where $\iota\colon \Lambda^3 \tilde{H}_{\mathbb{Z}_2} \subset \tilde{H}_{\mathbb{Z}_2} \otimes \Lambda^2 \tilde{H}_{\mathbb{Z}_2} \xrightarrow{\varpi \otimes \mathrm{id}} H_{\mathbb{Z}_2} \otimes \Lambda^2 \tilde{H}_{\mathbb{Z}_2}$.

Corollary

For $g \gg 0$,

$$c\colon \mathrm{tor}(\mathcal{I}(3)/\mathcal{I}(4)) \to \mathrm{tor}(Y_3\mathcal{IC}/Y_4)$$

is surjective.

Main Theorem [Nozaki-Suzuki-S. '25]

For every $n \geq 1$, there are elements of order 2 in $Y_{2n-1}\mathcal{IC}/Y_{2n}$ and in $\mathcal{I}(2n-1)/\mathcal{I}(2n)$ when $g \geq 3n$.

Proof

In the following pages, we see that there are elements of order 2 in $Y_{2n-1}\mathcal{IC}/Y_{2n}$.

Actually, we can check that there is some element $x \in \mathcal{I}(2n-1)/\mathcal{I}(2n)$ such that $c(x) \in Y_{2n-1}\mathcal{IC}/Y_{2n}$ is such an order 2 element. Since the Johnson homomorphism factors through $Y_{2n-1}\mathcal{IC}/Y_{2n}$,

$$x \in \mathrm{Ker}(\mathcal{I}(2n-1)/\mathcal{I}(2n) \to \mathcal{M}(2n-1)/\mathcal{M}(2n)).$$

Using Kupers and Randal-Williams's result, we see that $x \in \mathrm{tor}(\mathcal{I}(2n-1)/\mathcal{I}(2n))$.

Section 2. A functorial extension of the LMO invariant

the monoid of homology cylinders $\mathcal{IC}$

$=$ an extension of the Torelli group $\mathcal{I}$,

LMO functor $\tilde{Z}^Y$

$=$ a functorial extension of the LMO invariant Z^{LMO}.

$$\begin{array}{ccccc} & & \mathcal{S} & \xrightarrow[\log_\sqcup Z^{\mathsf{LMO}}]{} & \widehat{\mathcal{A}}^c(\varnothing) \\ & \nearrow & & & \\ \mathcal{I} & & & & \\ & \searrow & & & \\ & & \mathcal{IC} & \xrightarrow{\log_\sqcup \tilde{Z}^Y} & \widehat{\mathcal{A}}^c(\{1^\pm,\ldots,g^\pm\}). \end{array}$$

The LMO functor

The LMO functor

$$\tilde{Z}\colon \mathcal{LC}ob_q \to {}^{\mathsf{ts}}\widehat{\mathcal{A}}$$

is a functor, where

- $\mathcal{LC}ob_q$: some category of 3-dimensional cobordisms,
  Object: $\mathbb{Z}_{\geq 0}$ ($+$ parentheses),
  Morphism: cobordisms between $\Sigma_{g,1}$ to $\Sigma_{h,1}$ with some homological condition ($+$ parentheses),
- ${}^{\mathsf{ts}}\widehat{\mathcal{A}}$: the category of some Jacobi diagrams,
  Object: $\mathbb{Z}_{\geq 0}$,
  Morphism: $\mathbb{Q}$-series of Jacobi diagrams whose univalent vertices are colored by $\{1^+, \ldots, g^+, 1^-, \ldots, h^-\}$ with some condtion.

If we restrict $\tilde{Z}$ to $\mathcal{IC} \subset \mathcal{LC}ob_q(g, g)$, we obtain a monoid homomorphism

$$z := \log_{\sqcup} \tilde{Z}^Y : \mathcal{IC} \to \widehat{\mathcal{A}}^c \otimes \mathbb{Q}.$$

Theorem (Universality) [Cheptea-Habiro-Massuyeau '08]

The LMO functor $z := \log_{\sqcup} \tilde{Z}^Y : \mathcal{IC} \to \widehat{\mathcal{A}}^c \otimes \mathbb{Q}$ is universal among $\mathbb{Q}$-valued finite-type invariants of homology cylinders.

The LMO functor $z \colon \mathcal{IC} \to \hat{\mathcal{A}}^c \otimes \mathbb{Q}$ sends $Y_n\mathcal{IC}$ to $\hat{\mathcal{A}}^c_{\geq n} \otimes \mathbb{Q}$.

Let $z_n \colon \mathcal{IC} \xrightarrow{\log_{\sqcup} \tilde{Z}^Y} \widehat{\mathcal{A}}^c \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}^c_n \otimes \mathbb{Q}$ be the degree n part.

Theorem A [Cheptea-Habiro-Massuyeau '08] (Recall)

For $n \geq 1$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{A}^c_n & \xrightarrow{\mathfrak{s}_n} & Y_n\mathcal{IC}/Y_{n+1} \\ & {\scriptstyle \pm \mathrm{id} \otimes 1} \searrow & \downarrow {\scriptstyle z_n} \\ & & \mathcal{A}^c_n \otimes \mathbb{Q}. \end{array}$$

By Theorem A,

$$z_{n+1}(Y_{n+1}\mathcal{IC}) = (z_{n+1} \circ \mathfrak{s}_{n+1})(\mathcal{A}^c_{n+1}) = \mathrm{Im}(\mathcal{A}^c_{n+1} \otimes \mathbb{Z} \to \mathcal{A}^c_{n+1} \otimes \mathbb{Q}).$$

So the image is of integral coefficients, and it induces a map

$$\bar{z}_{n+1} := z_{n+1} \bmod \mathbb{Z} \colon \mathcal{IC}/Y_{n+1} \to \mathcal{A}^c_{n+1} \otimes \mathbb{Q}/\mathbb{Z},$$

and it restricts to a module homomorphism

$$\bar{z}_{n+1} \colon Y_n\mathcal{IC}/Y_{n+1} \to \mathcal{A}^c_{n+1} \otimes \mathbb{Q}/\mathbb{Z}.$$

Theorem B [Nozaki-Suzuki-S. '22]

For $n \geq 1$, the following diagram commutes:

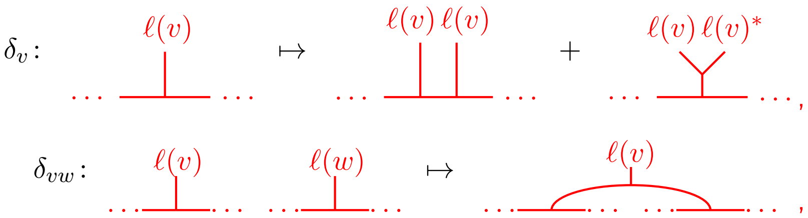
$$\begin{array}{ccc}
\mathcal{A}^c_n & \xrightarrow{\mathfrak{s}_n} & Y_n\mathcal{IC}/Y_{n+1} \\
\delta \big\downarrow & & \big\downarrow \bar{z}_{n+1} \\
\mathcal{A}^c_{n+1} \otimes \mathbb{Z}_2 & \xrightarrow[\mathrm{id} \otimes \frac{1}{2}]{} & \mathcal{A}^c_{n+1} \otimes \mathbb{Q}/\mathbb{Z}.
\end{array}$$

Definition ($\delta\colon \mathcal{A}_n^c \to \mathcal{A}_{n+1}^c \otimes \mathbb{Z}_2$)

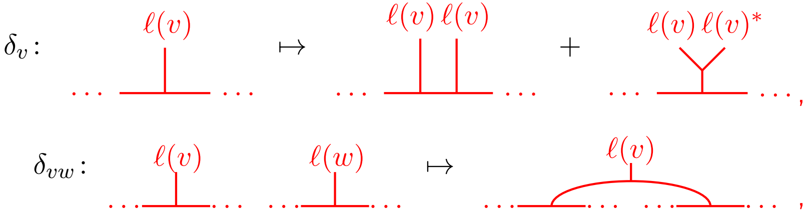
The homomorphism $\delta\colon \mathcal{A}_n^c \to \mathcal{A}_{n+1}^c \otimes \mathbb{Z}_2$ is defined by

$$\delta(J) = \sum_{v \in U(D)} \delta_v(J) + \sum_{\substack{v \neq w \in U(D) \\ l(v)=l(w)}} \delta_{vw}(J),$$

where $U(D) := \{\text{univalent vertices of } D\}$.



where $(j^+)^* := j^-$ and $(j^-)^* := j^+$.



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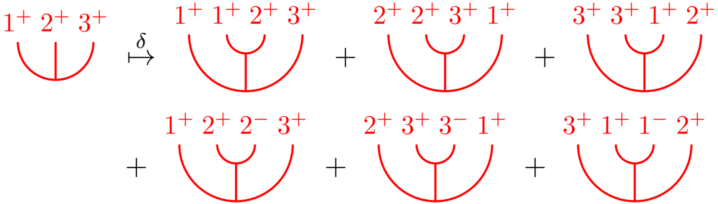
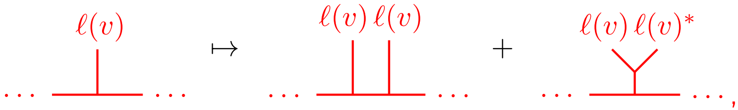
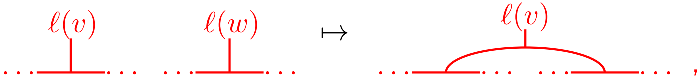
Example



δ_v :

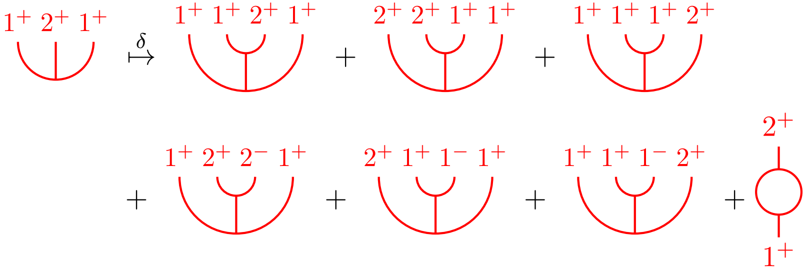


δ_{vw} :

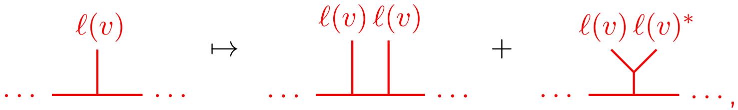


where $(j^+)^* := j^-$ and $(j^-)^* := j^+$.

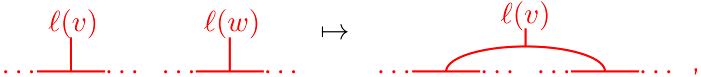
Example



δ_v :



δ_{vw} :



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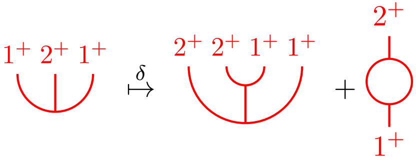
Example

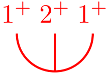


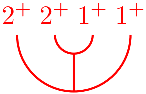
Theorem B [Nozaki-Suzuki-S. '22] (Recall)

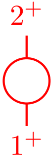
For $n \geq 1$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_n^c & \xrightarrow{\mathfrak{s}_n} & Y_n\mathcal{IC}/Y_{n+1} \\ \delta \downarrow & & \downarrow \bar{z}_{n+1} \\ \mathcal{A}_{n+1}^c \otimes \mathbb{Z}_2 & \xrightarrow[\mathrm{id}\otimes\frac{1}{2}]{} & \mathcal{A}_{n+1}^c \otimes \mathbb{Q}/\mathbb{Z}. \end{array}$$

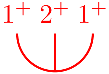
Example

$(\bar{z}_{n+1} \circ \mathfrak{s})\Bigg($ 

$\Bigg) = \dfrac{1}{2}$ 

$+ \dfrac{1}{2}$ 

$\neq 0.$

Especially, $\mathfrak{s}\Bigg($ 

$\Bigg) \in Y_n\mathcal{IC}/Y_{n+1}$ is a nontrivial element of order 2.

Furthermore, we also compute the part two degrees deeper in the LMO functor:

Theorem C [Nozaki-Suzuki-S. '25]

For $n \geq 1$, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{A}_n^c & \xrightarrow{\quad \mathfrak{s} \quad} & Y_n\mathcal{IC}/Y_{n+1} \\
{\scriptstyle \delta_2}\Big\downarrow & & \Big\downarrow{\scriptstyle \bar{\bar{z}}_{n+2}} \\
\mathcal{A}_{n+2}^c \otimes \mathbb{Z}_6 & \xrightarrow[\mathrm{id} \otimes \frac{1}{12}]{} & \mathcal{A}_{n+2}^c \otimes \mathbb{Q}/(\frac{1}{2}\mathbb{Z}).
\end{array}$$

Remark

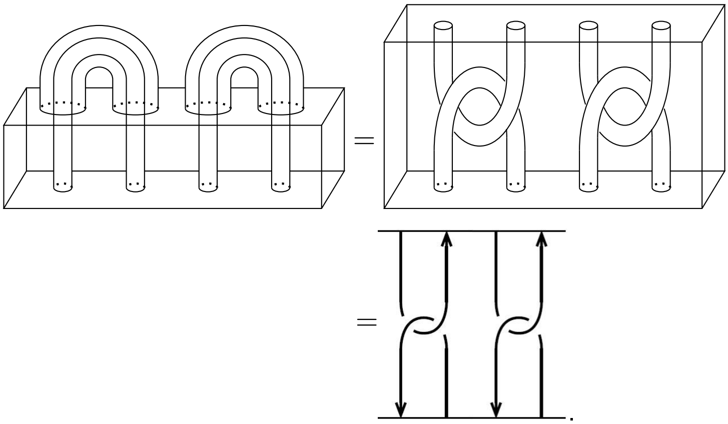
The part three or more degrees deeper may depend on the choice of Drinfeld associators, and is hard to compute.

Proof of Theorems A,B,C

The proofs of the previous 3 theorems are essentially the same.

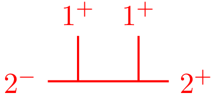
1. Let $J \in A_n^c$, and set the graph clasper $\mathfrak{s}(J)$ embedded in $\Sigma \times [-1,1]$ into some standard position.
2. The LMO functor $\tilde{Z}\colon \mathcal{LC}ob_q \to {}^{\mathsf{ts}}\widehat{\mathcal{A}}$ is a monoidal functor, i.e.
$$\tilde{Z}\left(\boxed{M \mid N}\right) = \tilde{Z}(M) \amalg \tilde{Z}(N)_{\mathsf{shift}}.$$
Hence, it suffices to compute the value of the LMO functor piecewisely.
3. Compute compositions of (the only leading terms of) Jacobi diagrams.

0. First, express the trivial homology cylinder $\Sigma \times [-1, 1]$ as a tangle.



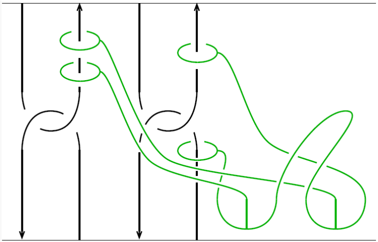
1. Set the graph clasper $\mathfrak{s}(J)$ embedded in $\Sigma \times [-1,1]$ into a standard position.

For example, let $J =$

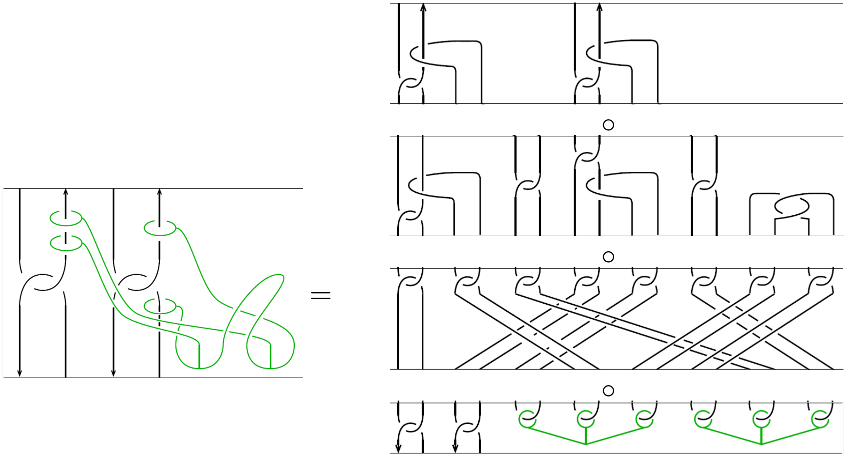


.

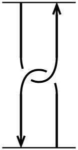
$s_n(J) =$



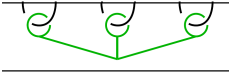
2. $\tilde{Z}$ is a monoidal functor. Hence, it suffices to compute the value of the LMO functor piecewisely.

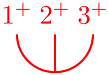


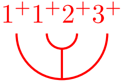
$\log_{\sqcup} \widetilde{Z}$ sends each piece of $\mathfrak{s}_n(J) \in Y_n\mathcal{IC}/Y_{n+1}$ as follows:

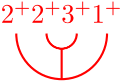
 $\mapsto$ 

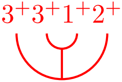
,

 $\mapsto$

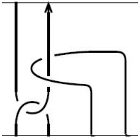
$-$ 

$+\frac{1}{2}$ 

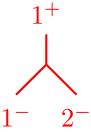
$+\frac{1}{2}$ 

$+\frac{1}{2}$ 

$+ (\mathrm{ideg} \geq 3)$,

 $\mapsto$ 

$+$ 

$-\frac{1}{2}$ 

$+ (\mathrm{ideg} \geq 2)$.

If we take the product of the leading terms, we see that

$$(z_n \circ \mathfrak{s}_n)(J) = \pm J.$$

Section 3 Results on the loop expansion of the LMO functor

We consider the n-loop part of $\log_{\sqcup} Z^K$ and $\log_{\sqcup} \tilde{Z}^Y$.

A string link is a generalization of a link in S^3.

Definition (String links)

Fix n distinct points $x_1, \ldots, x_n \in \operatorname{Int} D^2$. An n-component string link is a smooth, proper embedding

$$\sigma : \coprod_{i=1}^{n} [0,1] \longrightarrow D^2 \times [0,1]$$

such that $\sigma(i,0) = (x_i, 0)$ and $\sigma(i,1) = (x_i, 1)$ for $1 \leq i \leq n$.

Theorem [Habegger-Masbaum '00]

The leading term of the tree part of $\log_{\sqcup} Z^K$ over string links is equal to the Milnor invariant.

Theorem [Cheptea-Habiro-Massuyeau '08]

The leading term of the tree part of the LMO functor is the Johnson homomorphism. More precisely,

$$Y_n\mathcal{IC}/Y_{n+1} \xrightarrow{z_n} \mathcal{A}_n^c \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_{n,0}^c \otimes \mathbb{Q}$$

essencially coincides with the Johnson homomorphism.

Theorem [Massuyeau '12]

The whole tree part of the LMO functor essentially coincides with the total Johnson map

$$\tau^\theta \colon \mathcal{IC} \to \prod_{m=1}^{\infty} H_\mathbb{Q} \otimes \mathrm{Lie}_{m+1}(H_\mathbb{Q})$$

associated to some symplectic expansion θ.

Theorem [Bar-Natan, Garoufalidis '96]

Let K be an oriented framed knot.

The 1-loop part of $\log_{\sqcup} Z^K(K)$ is expressed by the Alexander polynomial of K.

	knot	homology cylinders
	The Kontsevich invariant	The LMO functor
0-loop	0 (Milnor invariants for string links)	Johnson homomorphisms
1-loop	Alexander polynomials	?
2-loop	$\mathbb{Z}$-equivariant Casson inv.?	?

Theorem [Milnor, Turaev, Goda-Sakasai, Friedl-Juhász-Rasmussen]

For link complements, closed oriented 3-manifolds M with $\dim H_1(M;Q) \geq 1$, and homology cylinders, the Alexander polynomial is the same as the Reidemeister-Turaev torsion with $Q(\mathbb{Z}[H_1(M)/\operatorname{tor}])$-coefficients.

Theorem[Nozaki-Suzuki-S. '23]

There is a crossed homo. $\tilde{\alpha}\colon \mathcal{IC} \to K_1(\widehat{\mathbb{Q}\pi})$ whose reduction $\mathcal{IC} \to \mathbb{Q}\pi/(I\pi)^{n+1}$ is a finite-type invariant of at most degree n.
Especially, it induces a crossed homo. $\alpha_n\colon \mathcal{IC}/Y_{n+1} \to K_1(\mathbb{Q}\pi/(I\pi)^{n+1})$.

Theorem [Nozaki-Suzuki-S. '23]

The leading term of the 1-loop part of the LMO functor is written by the Reidemeister-Turaev torsion

$$\tilde{\alpha}\colon \mathcal{IC} \to K_1(\widehat{\mathbb{Q}\pi}),$$

of pairs $(M, m_-(\Sigma))$ with $\widehat{\mathbb{Q}\pi}$-coefficient,
where $\widehat{\mathbb{Q}\pi}$ is the I-adic completion of $\mathbb{Q}\pi$, i.e. $\widehat{\mathbb{Q}\pi} = \varprojlim_n \dfrac{\mathbb{Q}\pi}{(I\pi)^n}$.

Proposition [Nozaki-Suzuki-S. '23]

1. $K_1(\widehat{\mathbb{Q}\pi}) \cong (\mathbb{Q}\pi)^{\times}_{\text{abel}} \cong \mathbb{Q}^{\times} \oplus \prod_{k=1}^{\infty}(H_{\mathbb{Q}}^{\otimes k}/(\text{cyclic}))$,
2. $K_1(\mathbb{Q}\pi/(I\pi)^{n+1}) \cong \mathbb{Q}^{\times} \oplus \prod_{k=1}^{n}(H_{\mathbb{Q}}^{\otimes k}/(\text{cyclic}))$.

Theorem [Nozaki-Suzuki-S. '23]

The map $\tilde{\alpha}\colon \mathcal{IC} \to K_1(\widehat{\mathbb{Q}\pi})$ is a crossed homo. whose reduction $\mathcal{IC} \to K_1(\mathbb{Q}\pi/(I\pi)^{n+1})$ is a finite-type invariant of at most degree n. Especially, it induces a crossed homo. $\alpha_n\colon \mathcal{IC}/Y_{n+1} \to K_1(\mathbb{Q}\pi/(I\pi)^{n+1})$.

Corollary

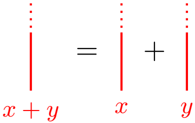
The induced map

$$\begin{array}{ccc} \tilde{\alpha}_n\colon Y_n\mathcal{IC}/Y_{n+1} & \to & \mathrm{Ker}(K_1(\mathbb{Q}\pi/(I\pi)^{n+1}) \to K_1(\mathbb{Q}\pi/(I\pi)^{n})) \\ & & \wr\| \\ & & H^{\otimes n}/(\text{cyclic}) \end{array}$$

is a homomorphism.

Definition (The module of Jacobi diagrams colored by $H_{\mathbb{Q}}$)

$$\mathcal{A}^c(H_{\mathbb{Q}}) := \frac{\mathbb{Q}\{\text{connected Jacobi diagrams colored by } H_{\mathbb{Q}}\}}{(\text{AS,IHX,multi-linear})}.$$

multi-linear:  for $x, y \in H_{\mathbb{Q}}$.

Prop [Habiro-Massuyeau '10]

There is an isomorphism

$$\kappa_n \colon \mathcal{A}_n^c \otimes \mathbb{Q} \cong \mathcal{A}_n^c(H_{\mathbb{Q}}),$$

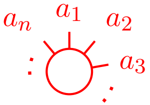
where $\kappa_n(J)$ is defined by changing labels from $i^- \mapsto \alpha_i \in H_{\mathbb{Q}}$, $i^+ \mapsto \beta_i$, and adding several terms (up to signs).

Denote by

$$Z_{n,1} := \text{1-loop part of } (\kappa_n \circ z_n) \colon Y_n\mathcal{IC}/Y_{n+1} \to \mathcal{A}_{n,1}^c(H_{\mathbb{Q}}).$$

Let

$$p_+ \colon H_{\mathbb{Q}}^{\otimes n}/(\text{cyclic}) \to \mathcal{A}^c_{n,1}(H_{\mathbb{Q}}),$$

$$a_1 \otimes \cdots \otimes a_n \mapsto$$ 

,

$$p_- \colon H_{\mathbb{Q}}^{\otimes n}/(\text{cyclic}) \to H_{\mathbb{Q}}^{\otimes n}/(\text{cyclic}),$$

$$a_1 \otimes \cdots \otimes a_n \mapsto a_1 \otimes \cdots \otimes a_n + (-1)^{n+1} a_1 \otimes \cdots \otimes a_n.$$

Theorem (Description of $\tilde{\alpha}_n$) [Nozaki-Suzuki-S. '23]

Our homomorphism $\tilde{\alpha}_n \colon Y_n\mathcal{IC}/Y_{n+1} \to H_{\mathbb{Q}}^{\otimes n}/(\text{cyclic})$ satisfies

1. $p_+ \circ \tilde{\alpha}_n = -2Z_{n,1} \colon Y_n\mathcal{IC}/Y_{n+1} \to \mathcal{A}^c_{n,1}(H_{\mathbb{Q}})$.
2. $p_- \circ \tilde{\alpha}_n = -\,\mathrm{ES} \circ \tau_n \colon Y_n\mathcal{IC}/Y_{n+1} \to H_{\mathbb{Q}}^{\otimes n}/(\text{cyclic})$,

where $\mathrm{ES} \colon H_{\mathbb{Q}} \otimes \mathrm{Lie}_{n+1}(H_{\mathbb{Q}}) \hookrightarrow H \otimes H_{\mathbb{Q}}^{\otimes(n+1)} \xrightarrow{C_{12}} H_{\mathbb{Q}}^{\otimes n}/(\text{cyclic})$,
and $C_{12} \colon H \otimes H_{\mathbb{Q}}^{\otimes(n+1)} \to H_{\mathbb{Q}}^{\otimes n}/(\text{cyclic})$ is defined by
$a \otimes (a_1 \otimes \cdots \otimes a_{n+1}) \mapsto \mu(a, a_1) a_2 \otimes \cdots \otimes a_{n+1}$.