On the degree-two part of the associated graded of the lower central series of the Torelli group

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joint work with Quentin Faes and Gwénaël Massuyeau

May 22, 2025

$g \geq 1$,

$\Sigma := \Sigma_{g,1}$: a compact oriented surface of genus g,

$\pi := \pi_1\Sigma$, $\pi = \Gamma_1\pi \supset \Gamma_2\pi \supset \cdots$: the lower central series,

$\mathcal{M}$: the mapping class group of Σ fixing $\partial\Sigma$ pointwise,

$\mathcal{I} := \mathrm{Ker}\Big(\mathcal{M} \to \mathrm{Aut}\Big(\dfrac{\pi}{\Gamma_2\pi}\Big)\Big)$: the Torelli subgroup.

Two filtrations of the Torelli group

$\mathcal{I} = \Gamma_1\mathcal{I} \supset \Gamma_2\mathcal{I} \supset \cdots$: the lower central series,

$\mathcal{I} = \mathcal{M}(1) \supset \mathcal{M}(2) \supset \cdots$: the Johnson filtration,

where $\mathcal{M}(k) = \mathrm{Ker}\Big(\mathcal{M} \to \Big(\mathrm{Aut}\dfrac{\pi}{\Gamma_{k+1}\pi}\Big)\Big)$.

Note that

1. $\Gamma_k\mathcal{I} \subset \mathcal{M}(k)$,
2. $\mathrm{Sp}(2g,\mathbb{Z}) \cong \mathcal{M}/\mathcal{I}$ acts on $\Gamma_k\mathcal{I}/\Gamma_{k+1}\mathcal{I}$ and $\mathcal{M}(k)/\mathcal{M}(k+1)$.

The monoid of homology cylinders

M: connected compact oriented 3-manifold with
$\partial M \cong \partial(\Sigma \times [-1, 1])$,
$m \colon \partial(\Sigma \times [-1, 1]) \xrightarrow{\cong} \partial M$: orientation-preserving and satisfies
$(m|_{\Sigma \times \{1\}})_* = (m|_{\Sigma \times \{-1\}})_* \colon H_*(\Sigma; \mathbb{Z}) \xrightarrow{\cong} H_*(M; \mathbb{Z})$,
$\mathcal{IC} = \{(M^3, m)\}$: the monoid of homology cylinders,
where $(M, m) \circ (N, n) := \begin{array}{|c|}\hline N \\ \hline M \\ \hline\end{array}$.

There is a series of equivalence relations $\sim_{Y_k}$ on $\mathcal{IC}$ for $k \geq 1$, called the Y_k-equivalence.

(i.e. $M \sim_{Y_k} N$ if and only if

N is obtained by surgeries along graph claspers with k-disks in M).

$Y_k\mathcal{IC} = \{(M, m) \in \mathcal{IC} \mid (M, m) \sim_{Y_k} (\Sigma \times [-1, 1], \mathrm{id})\}$.

$\mathcal{IC} = Y_1\mathcal{IC} \supset Y_2\mathcal{IC} \supset \cdots$: the Y-filtration.

There is a natural map $\mathfrak{c}\colon \mathcal{I} \to \mathcal{IC}$,

$\mathfrak{c}([f]) = (\Sigma \times [-1, 1], \mathrm{id}_{\text{bottom and side}} \cup f)$ satisfying $\mathfrak{c}(\Gamma_k\mathcal{I}) \subset Y_k\mathcal{IC}$,

and there is also an $\mathrm{Sp}(2g, \mathbb{Z})$-action on $Y_k\mathcal{IC}/Y_{k+1}$.

Problem

1. Determine the three $\mathrm{Sp}(2g,\mathbb{Z})$-modules

$$\bigoplus_{k\geq 1}\frac{\Gamma_k\mathcal{I}}{\Gamma_{k+1}\mathcal{I}},\ \bigoplus_{k\geq 1}\frac{\mathcal{M}(k)}{\mathcal{M}(k+1)},\ \bigoplus_{k\geq 1}\frac{Y_k\mathcal{IC}}{Y_{k+1}}.$$

2. Determine the kernels and images of the homomorphisms

$$\mathrm{inc}_*\colon \bigoplus_{k\geq 1}\frac{\Gamma_k\mathcal{I}}{\Gamma_{k+1}\mathcal{I}}\to\bigoplus_{k\geq 1}\frac{\mathcal{M}(k)}{\mathcal{M}(k+1)},$$

$$\mathfrak{c}_*\colon \bigoplus_{k\geq 1}\frac{\Gamma_k\mathcal{I}}{\Gamma_{k+1}\mathcal{I}}\to\bigoplus_{k\geq 1}\frac{Y_k\mathcal{IC}}{Y_{k+1}}.$$

My talk consists of 2 parts:

1. $\mathrm{tor}(\Gamma_2\mathcal{I}/\Gamma_3\mathcal{I}) = 0$,
2. the Sp-module str.'s of $\mathrm{tor}(Y_3\mathcal{IC}/Y_4)$ and $\mathrm{tor}(\Gamma_3\mathcal{I}/\Gamma_4\mathcal{I})$.

$H_R = H_1(\Sigma_{g,1}; R)$ for $R = \mathbb{Z}, \mathbb{Q}, \mathbb{Z}_2$.

Theorem [Johnson 1980, 1985]

Let $g \geq 3$. As $\mathrm{Sp}(2g, \mathbb{Z})$-modules,

$$\mathcal{I}/\Gamma_2\mathcal{I} \cong \Lambda^3 H_{\mathbb{Z}} \oplus_{\Lambda^3 H_{\mathbb{Z}_2}} B_3, \ \mathcal{I}/\mathcal{M}(2) \cong \Lambda^3 H_{\mathbb{Z}}.$$

Theorem [Hain, Morita]

Let $g \geq 6$. As $\mathrm{Sp}(2g, \mathbb{Q})$-modules,

$$\Gamma_2\mathcal{I}/\Gamma_3\mathcal{I} \otimes \mathbb{Q} \cong [2^2] + [1^2] + 2[0], \ \mathcal{M}(2)/\mathcal{M}(3) \otimes \mathbb{Q} \cong [2^2] + [1^2] + [0].$$

Theorem A [Faes-Massuyeau-S.]

When $g \geq 3$, $\Gamma_2\mathcal{I}/\Gamma_3\mathcal{I}$ is torsion-free.

Theorem [Cheptea-Habiro-Massuyeau 2008]

$$\mathcal{A}_n^{c,<}(H_{\mathbb{Q}}) \cong Y_n\mathcal{IC}/Y_{n+1} \otimes \mathbb{Q},$$

where $\mathcal{A}_n^{c,<}(H_{\mathbb{Q}})$ is the module of Jacobi diagrams with total orders.

Theorem [Nozaki-Suzuki-S. 2024]

For every $n \geq 1$,

there are elements of order 2 in $Y_{2n-1}\mathcal{IC}/Y_{2n}$ when $g \geq 3n$.

Theorem [Massuyeau-Meilhan 2003, 2013]

For $g \geq 3$,

$$\mathcal{IC}/Y_2 \cong \mathcal{I}/\Gamma_2\mathcal{I}\,(\cong \Lambda^3 H_{\mathbb{Z}} \oplus_{\Lambda^3 H_{\mathbb{Z}_2}} B_3),$$

and for $g \geq 1$,

$$Y_2\mathcal{IC}/Y_3 \cong \mathcal{A}_2^{c,<}(H_{\mathbb{Z}}) \quad \text{(torsion-free)}.$$

Theorem [Nozaki-Suzuki-S. 2022]

As an abelian group,

$$\mathrm{tor}(Y_3\mathcal{IC}/Y_4) \cong \mathrm{Lie}_3(H_{\mathbb{Z}_2}) \oplus S^2 H_{\mathbb{Z}_2}.$$

Let $\tilde{H}_{\mathbb{Z}_2} = H_1(U\Sigma; \mathbb{Z}_2)$ and $\varpi\colon \tilde{H}_{\mathbb{Z}_2} \to H_{\mathbb{Z}_2}$ be the induced homomorphism by the projection.

Theorem B [Faes-Massuyeau-S.]

As an $\mathrm{Sp}(2g, \mathbb{Z})$-module,

$$\mathrm{tor}(Y_3\mathcal{IC}/Y_4) \cong \frac{H_{\mathbb{Z}_2} \otimes \Lambda^2 \tilde{H}_{\mathbb{Z}_2}}{\iota(\Lambda^3 \tilde{H}_{\mathbb{Z}_2})},$$

where $\iota\colon \Lambda^3 \tilde{H}_{\mathbb{Z}_2} \subset \tilde{H}_{\mathbb{Z}_2} \otimes \Lambda^2 \tilde{H}_{\mathbb{Z}_2} \xrightarrow{\varpi \otimes \mathrm{id}} H_{\mathbb{Z}_2} \otimes \Lambda^2 \tilde{H}_{\mathbb{Z}_2}$.

1. $\mathrm{tor}(\Gamma_2\mathcal{I}/\Gamma_3\mathcal{I}) = 0$

Theorem [Faes-Massuyeau-S.]

When $g \geq 3$,

$$[\mathcal{I}, [\mathcal{I}, \mathcal{I}]] = [\mathcal{I}, \mathcal{M}(2)].$$

The proof is given by showing that the surjective map

$$\mathcal{I} \times \mathcal{M}(2) \to \frac{[\mathcal{I}, \mathcal{M}(2)]}{[\mathcal{I}, [\mathcal{I}, \mathcal{I}]]}, \ (\varphi, \psi) \mapsto [\varphi, \psi]$$

is the zero map.

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Taking a section of the projection $\mathcal{I}/\Gamma_2\mathcal{I} \to \mathcal{I}/\mathcal{M}(2) \cong \Lambda^3 H_{\mathbb{Z}}$, we have a Lie algebra homomorphism

$$\mathrm{Lie}\,\Lambda^3 H_{\mathbb{Z}} \cong \mathrm{Lie}(\mathcal{I}/\mathcal{M}(2)) \to \mathrm{Gr}^{\Gamma}\,\mathcal{I}.$$

By Theorem, its restriction to the $(\deg \geq 2)$-part

$$J\colon\ \mathrm{Lie}_{\geq 2}\,\Lambda^3 H_{\mathbb{Z}} = \mathrm{Lie}_{\geq 2}(\mathcal{I}/\mathcal{M}(2)) \to \mathrm{Gr}^{\Gamma}_{\geq 2}\,\mathcal{I}$$

is a canonical surjective Lie homomorphism.

We also denote $J^{\mathbb{Q}}\colon \mathrm{Lie}(\Lambda^3 H_{\mathbb{Q}}) \to \mathrm{Gr}^{\Gamma} \mathcal{I} \otimes \mathbb{Q}$.

Theorem (Hain 1997)

The ideal $\mathrm{Ker}\, J^{\mathbb{Q}}$ is generated by the degree 2 part for $g \geq 6$, and by the degree 2 and 3 parts for $g = 3, 4, 5$.

Question

$$\mathrm{Ker}\, J = \mathrm{Ker}\, J^{\mathbb{Q}} \cap \mathrm{Lie}(\Lambda^3 H_{\mathbb{Z}})?$$

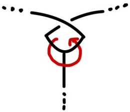
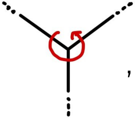
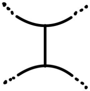
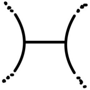
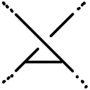
Yes, at least for the degree 2 part, i.e. $\mathrm{Ker}\, J_2 = \mathrm{Ker}\, J_2^{\mathbb{Q}} \cap \Lambda^2(\Lambda^3 H_{\mathbb{Z}})$.

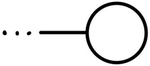
To show that $\Gamma_2\mathcal{I}/\Gamma_3\mathcal{I}$ is torsion-free, we compare
$J_2\colon \Lambda^2(\Lambda^3 H_{\mathbb{Z}}) \to \Gamma_2\mathcal{I}/\Gamma_3\mathcal{I}$ with another map
$B\colon \Lambda^2\mathcal{A}_1^c(H_{\mathbb{Q}}) \to \mathcal{A}_{2,0}^c(H_{\mathbb{Q}}) \oplus \mathcal{A}_{2,2}^c(H_{\mathbb{Q}})$.

Definition (modules of Jacobi diagrams)

For $R := \mathbb{Q}, \mathbb{Z}, \mathbb{Z}_2$, define

$$\mathcal{A}_n^c(H_R) = \frac{R\{\text{conn. Jacobi diagrams of } \deg = n \text{ colored by } H_R\}}{\text{AS, IHX, self-loop, multi-linear}}.$$

AS:  $= -$ , IHX:  $-$  $+$  $= 0$,

self-loop:  $= 0$, multi-linear: 

$=$ 

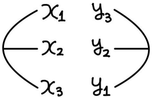
$+$ 

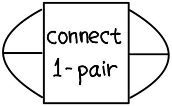
.

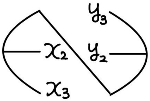
Note that $\mathcal{A}_1^c(H_{\mathbb{Q}}) \cong \Lambda^3 H_{\mathbb{Q}}$.

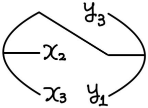
$B = (B^{(0)}, B^{(2)})\colon \Lambda^2 \mathcal{A}_1^c(H_{\mathbb{Q}}) \to \mathcal{A}_{2,0}^c(H_{\mathbb{Q}}) \oplus \mathcal{A}_{2,2}^c(H_{\mathbb{Q}})$

Define $B^{(0)}\colon \Lambda^2 \mathcal{A}_1^c(H_{\mathbb{Q}}) \to \mathcal{A}_{2,0}^c(H_{\mathbb{Q}})$,



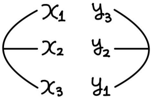
$\mapsto \sum$ (intersection number)


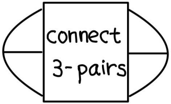
$= I(x_1, y_1)$


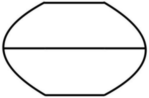
$+ I(x_1, y_2)$


$+ \cdots .$

Define $B^{(2)}\colon \Lambda^2 \mathcal{A}_1^c(H_{\mathbb{Q}}) \to \mathcal{A}_{2,2}^c(H_{\mathbb{Q}})$,



$\mapsto -\frac{1}{4} \sum$ (Π(intersection number))


$= -\frac{1}{4} I(x_1, y_3) I(x_2, y_2) I(x_3, y_1)$

$+ \cdots$

Theorem[Faes-Massuyeau-S.]

When $g \geq 3$, the kernel of the surjective homomorphism

$$J_2 \colon \Lambda^2(\Lambda^3 H_{\mathbb{Z}}) \to \Gamma_2\mathcal{I}/\Gamma_3\mathcal{I}$$

is equal to $\operatorname{Ker} B \cap \Lambda^2(\Lambda^3 H_{\mathbb{Z}})$.

The isom. $\Gamma_2\mathcal{I}/\Gamma_3\mathcal{I} \cong \Lambda^2(\Lambda^3 H_{\mathbb{Z}})/\operatorname{Ker} J_2 \cong B(\Lambda^2(\Lambda^3 H_{\mathbb{Z}}))$ implies:

Corollary

When $g \geq 3$, $\Gamma_2\mathcal{I}/\Gamma_3\mathcal{I}$ is torsion-free.

Corollary

When $g \geq 3$, there is an exact sequence

$$1 \longrightarrow \mathcal{I}/\Gamma_3\mathcal{I} \xrightarrow{\mathfrak{c}} \mathcal{IC}/Y_3 \xrightarrow{\alpha} S^2 H_{\mathbb{Z}} \xrightarrow{\text{cont. mod } 2} \mathbb{Z}_2 \longrightarrow 1.$$

Let $K := \mathrm{Ker}\, B \cap \Lambda^2(\Lambda^3 H_{\mathbb{Z}})$.

I show the sketch of the proof of $\mathrm{Ker}\, J_2 = K$.

Proposition

The following diagram commutes:

$$
\begin{array}{ccc}
\Lambda^2(\Lambda^3 H_{\mathbb{Q}}) & \xrightarrow{\ J_2^{\mathbb{Q}}\ } & \Gamma_2\mathcal{I}/\Gamma_3\mathcal{I} \otimes \mathbb{Q} \\
\Big\downarrow B & & \Big\downarrow (\tau_2, \text{Casson core}') \\
\mathcal{A}^c_{2,0}(H_{\mathbb{Q}}) \oplus \mathcal{A}^c_{2,2}(H_{\mathbb{Q}}) & = & \dfrac{S^2(\Lambda^2 H_{\mathbb{Q}})}{\Lambda^4 H_{\mathbb{Q}}} \oplus \mathbb{Q}.
\end{array}
$$

Hence, we have

$$\mathrm{Ker}\, J_2 \subset \mathrm{Ker}\, J_2^{\mathbb{Q}} \cap \Lambda^2(\Lambda^3 H_{\mathbb{Z}}) \subset \mathrm{Ker}\, B \cap \Lambda^2(\Lambda^3 H_{\mathbb{Z}}) = K.$$

Next, we give a sketch of the proof of $K \subset \mathrm{Ker}\, J_2$.

$S = \{a_1, b_1, \ldots, a_g, b_g\} \subset H_{\mathbb{Z}}$: a symplectic basis.
Denote $\tilde{a}_i := b_i$ and $\tilde{b}_i := a_i$.

Definition (elementary trivector)

For $s_1, s_2, s_3 \in S$,
we call $s_1 \wedge s_2 \wedge s_3 \in \Lambda^3 H_{\mathbb{Z}}$ an elementary trivector.

Definition (mixed contraction)

For a pair of elementary trivectors $s_1 \wedge s_2 \wedge s_3$, $s'_1 \wedge s'_2 \wedge s'_3$,
we call a pair $\{s_i, s'_j\}$ satisfying $s'_j = \tilde{s}_i$ a mixed contraction.

$\Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3) := (s_1 \wedge s_2 \wedge s_3) \wedge (s'_1 \wedge s'_2 \wedge s'_3) \in \Lambda^2(\Lambda^3 H_{\mathbb{Z}})$.
V_m: the submodule generated by $\Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3)$ with m mixed contractions,

$$\Lambda^2(\Lambda^3 H_{\mathbb{Z}}) = V_0 \oplus V_1 \oplus V_2 \oplus V_3.$$

Example

$\Lambda(a_1, a_2, a_3; b_1, a_3, a_4) \in V_1$, $\Lambda(a_1, a_2, b_2; b_1, b_2, b_3) \in V_2$,
$\Lambda(a_1, a_2, b_2; b_1, a_2, b_2) \in V_3$.

Definition (self-contraction)

For an elementary trivector $s_1 \wedge s_2 \wedge s_3$,
we call a pair $\{s_i, s_j\}$ satisfying $s_j = \tilde{s}_i$ a self-contraction.

$V_{1,n}$: the submodule generated by elements $\Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3)$ with

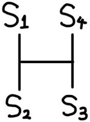
1. $\#\{\text{mixed cont. in } \Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3)\} = 1$,
2. $\#\{\text{self. cont. in } s_1 \wedge s_2 \wedge s_3\} + \#\{\text{self. cont. in } s'_1 \wedge s'_2 \wedge s'_3\} = n$.

The module V_1 decomposes into the direct sum

$$V_1 = V_{1,0} \oplus V_{1,1} \oplus V_{1,2}.$$

Example

$\Lambda(a_1, a_2, a_3; b_1, a_2, a_3) \in V_{1,0}$, $\Lambda(a_1, a_2, b_2; b_1, b_3, b_4) \in V_{1,1}$,
$\Lambda(a_1, a_2, b_2; b_1, a_3, b_3) \in V_{1,2}$.

W_m: the submodule generated by 

$\in \mathcal{A}^c_{2,0}(H_{\mathbb{Q}})$ such that $\#\{\{i,j\} \mid s_i = \tilde{s}_j\} = m$. Then, we have a decomposiition

$$\mathcal{A}^c_{2,0}(H_{\mathbb{Q}}) = W_0 \oplus W_1 \oplus W_2.$$

The map $B^{(0)}$ preserves the direct sum decomposition as

$$B^{(0)}(V_0) = 0, \qquad B^{(0)}(V_{1,0}) \subset W_0,$$
$$B^{(0)}(V_{1,1} \oplus V_2) \subset W_1, \qquad B^{(0)}(V_{1,2} \oplus V_3) \subset W_2.$$

Hence, we have:

Lemma

$$K = V_0 \oplus (K \cap V_{1,0}) \oplus (K \cap (V_{1,1} \oplus V_2)) \oplus (K \cap (V_{1,2} \oplus V_3)).$$

Let $G = \langle \{E_i\}_{i=1}^g, \{F_{ij}\}_{1\leq i<j\leq g} \rangle \subset \mathrm{Sp}(2g,\mathbb{Z})$, where

- $E_i(a_i) = -b_i$, $E_i(b_i) = a_i$, and E_i fix the other elements of S,
- $F_{ij}(a_i) = a_j$, $F_{ij}(b_i) = b_j$, $F_{ij}(a_j) = a_i$, $F_{ij}(b_j) = b_i$, and F_{ij} fix the other elements of S.

Note that $G \cong \mathbb{Z}_4^g \rtimes \mathfrak{S}_g$ "acts" on the set of elementary trivectors up to signs.

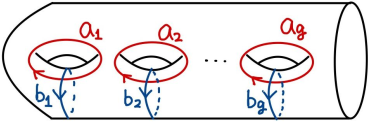
Lemma

As a G-submodule of $\Lambda^2(\Lambda^3 H_{\mathbb{Z}})$, V_0 is generated by

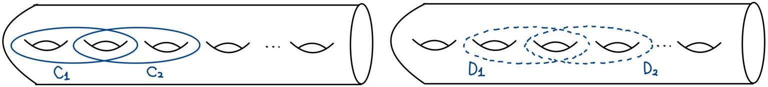
$$\Lambda(a_1,a_2,a_3;a_4,a_5,a_6), \Lambda(a_1,b_1,a_2;a_3,a_4,a_5), \Lambda(a_1,b_1,a_2;a_4,b_3,a_4),$$
$$\Lambda(a_1,a_2,a_3;a_3,a_4,a_5), \Lambda(a_1,a_2,a_3;a_2,a_3,a_4), \Lambda(a_1,b_1,a_2;a_2,a_3,b_3),$$
$$\Lambda(a_1,a_2,a_3;a_2,a_3,a_4).$$

Proof that $\Lambda(a_1, a_2, a_3; a_2, a_3, a_4) \in \operatorname{Ker} J_2$

Choose a homology basis as follows:



Consider Dehn twists along 4 simple closed curves



We can easily check that $[C_1, C_2] = C_1 C_2 C_1^{-1} C_2^{-1}, [D_1, D_2] \in \mathcal{I}$ and

$$\tau_1([C_1, C_2]) = \pm a_1 \wedge a_2 \wedge a_3, \;\; \tau_1([D_1, D_2]) = \pm a_2 \wedge a_3 \wedge a_4.$$

Hence, $J_2 \colon \Lambda^2(\Lambda^3 H_{\mathbb{Z}}) \to \Gamma_2\mathcal{I}/\Gamma_3\mathcal{I}$ maps

$$J_2(\Lambda(a_1, a_2, a_3; a_2, a_3, a_4)) = [[C_1, C_2], [D_1, D_2]] = 1 \in \Gamma_2\mathcal{I}/\Gamma_3\mathcal{I}.$$

Lemma

As a G-submodlule of $\Lambda^2(\Lambda^3 H_{\mathbb{Z}})$, $K \cap V_{1,0}$ is generated by

$$\mathrm{D}(a_1, a_2; a_5, a_3; a_3, a_4), \mathrm{D}(a_1, a_2; a_3, a_4; a_3, a_4), \mathrm{D}(a_1, a_2; a_3, b_1; a_3, a_4),$$
$$\mathrm{D}(a_3, a_1; a_4, a_2; a_2, a_3), \mathrm{D}(a_3, a_1; b_1, a_2; a_2, a_3), \mathrm{D}(a_1, a_2; a_4, a_3; a_2, a_1),$$
$$\mathrm{IHX}_1(a_1; a_2, a_3, a_4; b_1),$$

where $\mathrm{D}(x, y, c_1; c_2, x', y')$
$$= I(c_1, \tilde{c}_1)\Lambda(x, y, c_1; \tilde{c}_1, x', y') - I(c_2, \tilde{c}_2)\Lambda(x, y, c_2; \tilde{c}_2, x'y'),$$
$$\mathrm{IHX}_1(a_1; a_2, a_3, a_4; b_1) = \Lambda(a_1, a_2, b_1; a_1, a_3, a_4)$$
$$+ \Lambda(a_1, a_3, b_1; a_1, a_3, a_2) + \Lambda(a_1, a_4, b_1; a_1, a_2, a_3).$$

Theorem [Gervais-Habegger 2002]

$$\mathrm{IHX}_1(a_1; a_2, a_3, a_4; b_1) \in \mathrm{Ker}\, J_2$$

Theorem [Faes-Massuyeau-S.]

As a G-submodule of $\Lambda^2(\Lambda^3 H_{\mathbb{Z}})$, K is generated by 26 elements, and they are in $\mathrm{Ker}\, J_2$.

2. The Sp-module str.'s of $\mathrm{tor}(Y_3\mathcal{IC}/Y_4)$ and $\mathrm{tor}(\Gamma_3\mathcal{I}/\Gamma_4\mathcal{I})$

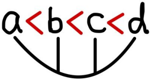
In this part, we denote $H := H_{\mathbb{Z}_2} = H_1(\Sigma; \mathbb{Z}_2)$.

Definition (modules of Jacobi diagrams with total orders)

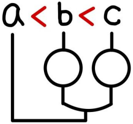
For a ring $R = \mathbb{Z}, \mathbb{Z}_2, \mathbb{Q}$, define

$$\mathcal{A}_n^{c,<}(H_R) = \frac{R\left\{\begin{array}{c}\text{conn. Jacobi diagrams } D \text{ of } \deg = n \text{ colored by } H_R \\ \text{with total orders on the set } U(D) \text{ of univ. vert's}\end{array}\right\}}{\text{AS}, \text{IHX}, \text{self-loop}, \text{STU-like}, \text{multi-linear}}$$

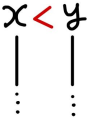
Example

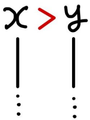


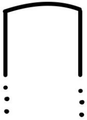
$\in \mathcal{A}_2^{c,<}(H_R)$,



$\in \mathcal{A}_5^{c,<}(H_R)$,

STU-like relation:


$-$


$= I(x, y)$


The Sp-equivariant surgery map

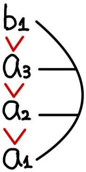
Theorem [Habiro, Goussarov]

When $n \geq 2$, there is a natural Sp-equivariant surjective homomorphism

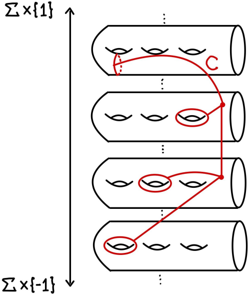
$$\mathfrak{s}\colon \mathcal{A}_n^{c,<}(H_{\mathbb{Z}}) \to Y_n\mathcal{IC}/Y_{n+1}$$

defined by surgeries along claspers.

Example



$\mapsto$



$= (\Sigma \times [-1,1])_C$

Theorem [Nozaki-Suzuki-S. 2022]

Let $g \geq 1$. As abelian groups, we have isomorphisms

1. $\mathrm{tor}\,\mathcal{A}_3^{c,<}(H_\mathbb{Z}) \cong (H \otimes \Lambda^2 H) \oplus H^{\otimes 2}$.
2. $\mathrm{tor}(Y_3\mathcal{IC}/Y_4) \cong \mathrm{Lie}_3(H) \oplus S^2 H$.

$\tilde{H} := H_1(U\Sigma_{g,1};\mathbb{Z}_2)$.

Denote by ι the composition $\Lambda^3\tilde{H} \subset \tilde{H} \otimes \Lambda^2\tilde{H} \xrightarrow{\varpi\otimes\mathrm{id}} H \otimes \Lambda^2\tilde{H}$, which is injective.

Theorem [Faes-Massuyeau-S.]

Let $g \geq 1$. As $\mathrm{Sp}(2g,\mathbb{Z})$-modules, we have isomorphisms

1. $\mathrm{tor}\,\mathcal{A}_3^{c,<}(H_\mathbb{Z}) \cong H \otimes \Lambda^2\tilde{H}$,
2. $\mathrm{tor}(Y_3\mathcal{IC}/Y_4) \cong \dfrac{H \otimes \Lambda^2\tilde{H}}{\iota(\Lambda^3\tilde{H})}$.

Corollary

When $g \geq 5$, the natural homomorphism

$$\mathfrak{c}\colon \operatorname{tor}(\Gamma_3\mathcal{I}/\Gamma_4\mathcal{I}) \to \operatorname{tor}(Y_3\mathcal{IC}/Y_4)$$

is surjective.

We do not know whether the above map is an isomorphism.

$\mathcal{IH} = \mathcal{IC}/$(4-dim. homology cob.): the homology cobordism group,
$\mathcal{IH} = Y_1\mathcal{IH} \supset Y_2\mathcal{IH} \supset Y_3\mathcal{IH} \supset \cdots$: the Y-filtration.

The surgery map induces the Sp-equivariant surjective homomorphism

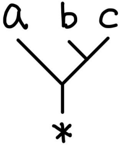
$$\mathfrak{s}\colon \mathcal{A}^c_{n,0}(H_{\mathbb{Z}}) \to Y_n\mathcal{IH}/Y_{n+1}$$

Levine, Conant, Schneidermann, and Teichner studied the module $\mathcal{A}^c_{n,0}(H_{\mathbb{Z}})$, especially 2-torsions.
Our result can be seen as a "lift" of their result.

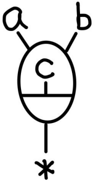
For $R = \mathbb{Z}, \mathbb{Q}, \mathbb{Z}_2$, define

$$\mathcal{A}_n^{c,*}(H_R) = \frac{R\left\{ \begin{array}{c} \text{connected rooted Jacobi diagrams of } \deg = n \\ \text{colored with } H_R \end{array} \right\}}{\text{AS, IHX, self-loop, multli-linear}}.$$

Example



$\in \mathcal{A}_{2,0}^{c,*}(H_R)$,

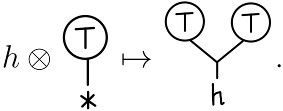


$\in \mathcal{A}_{6,2}^{c,*}(H_R)$.

2-torsions in $\mathcal{A}^c_{2n+1,0}(H_\mathbb{Z})$

Levine constructed an Sp-equiv. homomorphism

$$H_\mathbb{Z} \otimes \mathcal{A}^{c,*}_{n,0}(H_\mathbb{Z}) \to \mathcal{A}^c_{2n+1,0}(H_\mathbb{Z})$$



All elements in the image are of order 2, and the map induces $\mathrm{sq}\colon H \otimes \mathcal{A}^{c,*}_{n,0}(H) \to \mathrm{tor}\,\mathcal{A}^c_{2n+1,0}(H_\mathbb{Z})$.

Proposition [Conant-Schneidermann-Teichner 2012]

For $n \geq 0$, $\mathrm{sq}\colon H \otimes \mathcal{A}^{c,*}_{n,0}(H) \to \mathrm{tor}\,\mathcal{A}^c_{2n+1,0}(H_\mathbb{Z})$ is surjective.
Moreover, it induces an isomorphism

$$H \otimes \mathrm{Lie}_{n+1}(H) \cong \mathrm{tor}\,\mathcal{A}^c_{2n+1,0}(H_\mathbb{Z}).$$

We want to lift the map to the one whose target is $\mathcal{A}^{c,<}_{2n+1}(H_{\mathbb{Z}})$.

Proposition

There exists a well-defined Sp-equivariant homomorphism

$$E\colon H \otimes \overset{\bullet}{\mathcal{A}}{}^{c,<,*}_{n}(\tilde{H}) \to \mathrm{tor}\, \mathcal{A}^{c,<}_{2n+1}(H_{\mathbb{Z}})$$

such that the diagram

$$\begin{array}{ccc} H \otimes \overset{\bullet}{\mathcal{A}}{}^{c,<,*}_{n}(\tilde{H}) & \xrightarrow{\ E\ } & \mathrm{tor}\, \mathcal{A}^{c,<}_{2n+1}(H_{\mathbb{Z}}) \\ {\scriptstyle \text{proj.}\circ\text{forget}\circ(\mathrm{id}_H \otimes \varpi)}\downarrow & & \downarrow{\scriptstyle \text{proj.}\circ\text{forget}} \\ H \otimes \mathcal{A}^{c,*}_{n,0}(H) & \underset{\mathrm{sq}}{\twoheadrightarrow} & \mathrm{tor}\, \mathcal{A}^{c}_{2n+1,0}(H_{\mathbb{Z}}) \end{array}$$

commutes.

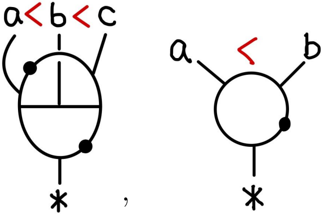
Let J be a connected rooted Jacobi diagrams colored by $\tilde{H}$ with

- n trivalent vertices,
- a total order on the set of univalent vertices,
- beads in (interior of) edges,

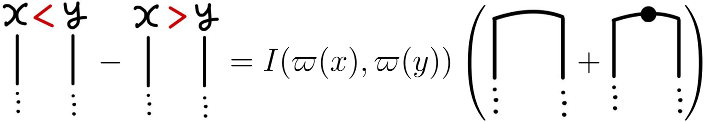
Definition(module of Jacobi diagrams with beads)

$$\dot{\mathcal{A}}_n^{c,<,*}(\tilde{H}) = \frac{\mathbb{Z}_2\{\text{Jacobi diagrams } J \text{ as above}\}}{\text{AS, IHX, self-loop, multi-linear, BSTU-like, B}}.$$

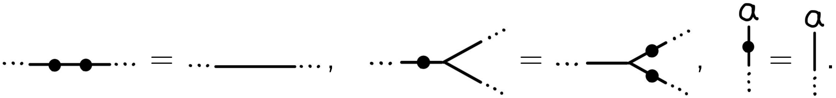
Example



BSTU-like:



B:

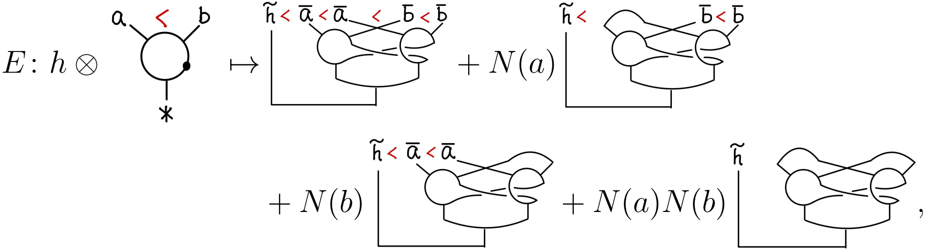


Proposition

There is a well-defined group homomorphism

$$E\colon H \otimes \mathring{\mathcal{A}}_n^{c,<,*}(\tilde{H}) \to \operatorname{tor} \mathcal{A}_{2n+1}^{c,<}(H_{\mathbb{Z}})$$

defined by $E(h \otimes D) = \displaystyle\sum_{S \subset U(D)} \left(\prod_{s \in S} N(\mathrm{col}(s)) \right) \widetilde{D}(h, S)$.



where $\tilde{h}, \bar{a}, \bar{b} \in H_{\mathbb{Z}}$ are lifts of $h, \varpi(a), \varpi(b) \in H$, respectively.

$N\colon \tilde{H} \to \mathbb{Z}_2$ and Johnson's canonical lifting of H

Let z denote the generator of $\mathrm{Ker}(\tilde{H} \to H) \cong \mathbb{Z}_2$.

For $x \in H$, choose a collection of disjoint smooth scc's $\{\alpha_i\}_{i=1}^m$ such that $x = \sum_{i=1}^m \alpha_i \in H$. Then, define

$$\tilde{x} = \sum_{i=1}^{m} \vec{\alpha_i} + mz \in \tilde{H},$$

where $\vec{\alpha_i} \in \tilde{H}$ are represented by smooth oriented loops in $\Sigma_{g,1}$.

Theorem[Johnson 1980]

The element $\tilde{\alpha} \in \tilde{H}$ only depends on $x \in H$.

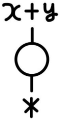
In other words, $H \to \tilde{H},\ \alpha \mapsto \tilde{\alpha}$ is well-defined and Sp-equivariant.

Let $N\colon \tilde{H} \to \mathbb{Z}_2$ be the Sp-equiv. map (not a homo.) defined by

$$N(x) = x - \widetilde{\varpi(x)} \in \mathbb{Z}_2 \cong \mathrm{Ker}(\tilde{H} \to H).$$

Proof of Proposition

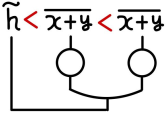
We show that E respects the multi-linear relation. Let

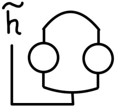
$D_{x+y} =$ 

, $D_x =$ 

, $D_y =$ 

$\in \dot{\mathcal{A}}_2^{c,<,*}(\tilde{H})$.

Then, $E(h \otimes D_{x+y}) =$ 

$+ N(x+y)$ 

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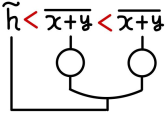
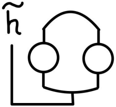
$$E(h \otimes D_{x+y}) = $$ 

$$+ N(x+y)$$ 

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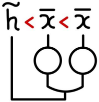
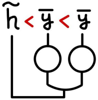
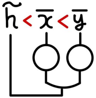
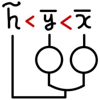
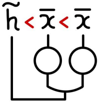
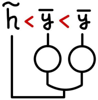
1st term $=$ 

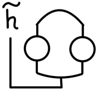
$+$ 

$+$ 

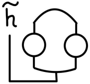
$+$ 

$=$ 

$+$ 

$+ I(\bar{x}, \bar{y})$ 

$\in \mathcal{A}_5^{c,<}(H_{\mathbb{Z}})$,

2nd term $= \{N(x) + N(y) + I(\bar{x}, \bar{y})\}$ 

,

where we used the equality $N(x+y) = N(x) + N(y) + I(\bar{x}, \bar{y})$ shown by Johnson.

Thus, we have $E(h \otimes D_{x+y}) = E(h \otimes D_x) + E(h \otimes D_y)$.