

On the LMO functor constructed by Cheptea, Habiro and Massuyeau

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Contents

① Notation

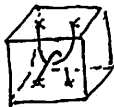
② q -tangles in $[-1, 1]^3$ $\xrightarrow{\text{Kontsevich int. } \hat{\mathbb{Z}}}$ Jacobi diagrams based on 1-mfd
 $\downarrow$ generalize

q -tangles in homology cubes $\xrightarrow{\text{LMO-Kontsevich integral } \mathbb{Z}}$

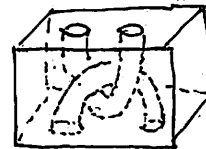
③ bottom-top tangles in homology cubes

$\begin{matrix} D \\ \sim \\ = \end{matrix}$

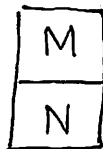
Lagrangian cobordisms between cpt surfaces



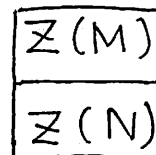
$\downarrow$ ② ③
not homo.



Lagrangian q -cobordisms $\xrightarrow{\mathbb{Z} \cdot D^{-1}}$ Jacobi diagrams based on 1-mfd



$\mapsto \otimes \neq$



$\downarrow$ modify

④ Lagrangian q -cobordisms $\xrightarrow[\text{homo}]{\text{monoid } \hat{\mathbb{Z}}}$ top-substantial Jacobi diagrams

⑤ $(\underbrace{I(F_q)}_{\text{Torelli grp}}) \mathcal{L} \text{Cob}_q(q, q) \xrightarrow{\hat{\mathbb{Z}}} \text{tsA}(q, q)$

tree part $\swarrow$

Johnson homo.

$\searrow$ loop part

finite type inv.
e.g. Casson inv.

Reference

[CHM]: D. Cheptea, H. Habiro, G. Massuyeau, "A functorial LMO invariant for Lagrangian Cobordisms", *Geom. Topol.* 12 (2008), 1091-1170

[BGRT1]: D. Bar-Natan, S. Garoufalidis, L. Rozansky, D.P. Thurston, "The Århus integral of rational homology 3-spheres I", *Selecta Math. (N.S.)* 8 (2002), 315-339

[BGRT2]: D. Bar-Natan, S. Garoufalidis, L. Rozansky, D.P. Thurston, "The Århus integral of rational homology 3-spheres II", *Selecta Math. (N.S.)* (2002), 341-371

[CDM]: S. Chmutov, S. Duzhin, J. Mostovoy, "Introduction to Vassiliev knot invariants", Cambridge University Press, 2012

[JMM]: D.M. Jackson, I. Moffatt, A. Morales, "On the group-like behavior of the Le-Murakami-Ohtsuki invariant", *J. Knot Theory Ramifications* 16 (2007), 699-718

[LM]: Representation of the category of tangles by Kontsevich's iterated integral, *Comm. Math. Phys.* 168 (1995), 535-562

[O]: T. Ohtsuki, "Quantum invariants", *Series on Knots and Everything* 29, World Scientific Publishing Co. (2002)

§ Jacobi diagrams and operations

X : cpt oriented 1-mfd,

C : finite set.

Jacobi diagrams based on (X, C)

D : Jacobi diagram based on (X, C)

$\Leftrightarrow$ D is a uni-trivalent graph
def

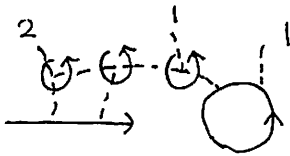
whose univalent vertices are either embedded into X

or colored with elements of C

and trivalent vertices are oriented.

Example

$$X = \rightarrow \sqcup \bigcirc, \quad C = \{1, 2\}.$$



$A(X, C)$

$A(X, C) = \frac{\mathbb{Q} \{ \text{Jacobi diagrams based on } (X, C) \}}{AS, IHX, STU}$; the space of Jacobi diagrams.

AS

$$\bigcirc = - \bigcirc$$

IHX

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} + \begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array} = 0$$

STU

$$\begin{array}{c} \parallel \\ \parallel \end{array} - \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} - \begin{array}{c} \parallel \\ \parallel \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \right)$$

Example

$$\begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} = \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} = 2 \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} = - \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array}$$

②

$$\begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} \stackrel{\text{IHX}}{=} \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} - \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} \stackrel{\text{AS}}{=} 2 \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array}$$

$$\begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} \stackrel{\text{STU}}{=} \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} - \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} \stackrel{\text{AS}}{=} -2 \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array}$$

Lemma (see [O] Prop 6.1)

D_1 : a Jacobi diagram based on $(X \uparrow, c)$,

D_2 : based on $\uparrow$.

Then, in $A(X \sqcup \uparrow, c)$,

$$\begin{array}{c} \boxed{D_1} \\ | \\ \boxed{D_2} \end{array} \times \uparrow = \begin{array}{c} \boxed{D_2} \\ | \\ \boxed{D_1} \end{array} \uparrow$$

Example

$$\begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} \stackrel{\text{STU}}{=} \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} + \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} \stackrel{\text{IHX}}{=} \dots = \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array} + \begin{array}{c} \text{---} \circ \text{---} \\ | \\ c \end{array}$$

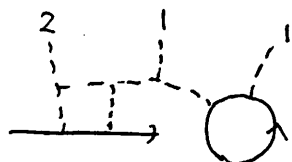
degree of Jacobi diagrams

$i\text{-deg} = \# \text{ trivalent vertices,}$

$e\text{-deg} = \# \text{ univalent vertices,}$

$$\text{deg} = \frac{(i\text{-deg}) + (e\text{-deg})}{2}.$$

example



$$i\text{-deg} = 4$$

$$e\text{-deg} = 8$$

$$\text{deg} = \frac{4+8}{2} = 6.$$

Rem

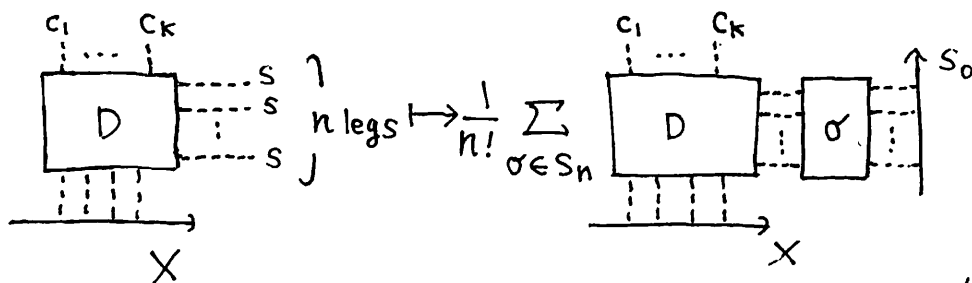
The AS and IHX relations preserve $i\text{-deg}$, $e\text{-deg}$ and deg .

The STU relation preserves deg .

In the following, we consider the degree completion of $\mathcal{A}(X, C)$, and denote it also by $\mathcal{A}(X, C)$.

$$\underline{\chi_S : \mathcal{A}(X, C \sqcup \{s\}) \rightarrow \mathcal{A}(X \uparrow^s, C)}$$

$$\mathcal{A}(X, C \sqcup \{s\}) \xrightarrow{\chi_S} \mathcal{A}(X \uparrow^s, C)$$



where $\boxed{(12)} = \text{crossing}$, $\boxed{(123)} = \text{triple crossing}$, and so on.

For $S = \{s_1, s_2, \dots, s_n\}$, we denote $\chi_S = \chi_{s_1} \circ \chi_{s_2} \circ \dots \circ \chi_{s_n}$.

Lemma

Let D be a Jacobi diagram based on $(X \uparrow^S, c)$ with n -legs attached to $\uparrow^S$.

$$D = \text{[Diagram: A rectangle with } k \text{ legs } c_1, c_2, \dots, c_k \text{ on top and } n \text{ legs on the right. The right side is labeled } \uparrow^S \text{ and } n\text{-legs. The bottom is labeled } X \text{ with an arrow pointing right.}]$$

$$\sigma D = \text{[Diagram: A rectangle with } k \text{ legs } c_1, c_2, \dots, c_k \text{ on top and } n \text{ legs on the right. The right side is labeled } \uparrow^S \text{ and } (\sigma \in S_n). \text{ A box labeled } \sigma \text{ is on the right side. The bottom is labeled } X \text{ with an arrow pointing right.}]$$

$$\tilde{D} = \text{[Diagram: A rectangle with } k \text{ legs } c_1, c_2, \dots, c_k \text{ on top and } n \text{ legs on the right. The right side is labeled } \uparrow^S \text{ and } S. \text{ The bottom is labeled } X \text{ with an arrow pointing right.}]$$

(1) For $\sigma \in S_n$, $D - \sigma D \in \mathcal{A}(X \uparrow^S, c)$ is represented by a sum $\Gamma_\sigma(D)$ of Jacobi diagrams with at most $(n-1)$ -legs attached to $\uparrow^S$.

(2) Define, for a Jacobi diagram D with at most n -legs are attached to $\uparrow^S$,

$$\tau_n(D) = \begin{cases} \tilde{D} + \frac{1}{n!} \sum_{\sigma \in S_n} \tau_{n-1}(\Gamma_\sigma(D)) \in \mathcal{A}(X, C \sqcup \{s\}) & \text{if } \# \text{ attached legs} = n, \\ \tau_{n-1}(D) & \text{if } < n, \end{cases}$$

inductively, $(\tau_1(D) = \tilde{D})$.

Then, $\tau_n(D)$ does not depend on the choice of $\Gamma_\sigma(D)$,

and $\tau: \mathcal{A}(X \uparrow^S, c) \rightarrow \mathcal{A}(X, C \sqcup \{s\})$ defined by

$\tau|_{\# \text{ attached legs} \leq n} = \tau_n$ is the inverse map of

$$\chi_s: \mathcal{A}(X, C \sqcup \{s\}) \rightarrow \mathcal{A}(X \uparrow^S, c).$$

Rem

We can also define $\chi_s: \mathcal{A}(X, C \sqcup \{s\}) \rightarrow \mathcal{A}(X \odot^S, c)$,

but it is not an isomorphism. Instead, we have.

$$\begin{array}{ccc} \mathcal{A}(X, C \sqcup \{s\}) & \xrightarrow{\chi_s} & \mathcal{A}(X \uparrow^S, c) \\ \downarrow & & \downarrow \\ \frac{\mathcal{A}(X, C \sqcup \{s\})}{s\text{-link relation}} & \cong & \mathcal{A}(X \odot^S, c). \end{array}$$

Proof [CDM, § 5.7.1](1) When $\sigma = (12)$,

$$D - \sigma D = \begin{array}{c} \text{Diagram 1} \end{array} - \begin{array}{c} \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array} = \Gamma_{(12)}(D)$$

For general $\sigma \in S_n$, σ can be written as a product $\sigma_1 \dots \sigma_n$ of $\{(i \ i+1) \mid i=1, \dots, n-1\}$.Then, $D - \sigma D = (D - \sigma_1 D) + \sigma_1 (D - \sigma_2 D) + \dots + \sigma_1 \dots \sigma_{n-1} (D - \sigma_n D)$.(2) For welldefinedness, see CDM. For a Jacobi diagram D with n -legs attached,

$$\begin{aligned} (\chi \circ \tau)(D) &= \chi\left(\tilde{D} + \frac{1}{n!} \sum_{\sigma \in S_n} \tau_{n-1}(\Gamma_{\sigma}(D))\right) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \sigma D + \frac{1}{n!} \sum_{\sigma \in S_n} \underbrace{(\chi \circ \tau)(\Gamma_{\sigma}(D))}_{\substack{\parallel \\ \Gamma_{\sigma}(D) \text{ induction}}} \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \sigma D + \frac{1}{n!} \sum_{\sigma \in S_n} (D - \sigma D) \\ &= D. // \end{aligned}$$

$$\begin{aligned} \therefore \chi_{\{x\}}^{-1}(\text{III}) &= \tau_3(\text{III}) \\ &= \text{III} + \frac{1}{6} \tau_2 \left(2 \text{II} + \frac{5}{2} \text{I} - 3 \text{X} + \text{Y} \right) \\ &= \text{III} + \frac{1}{6} \left(2 \text{II} - 2 \text{Y} + \frac{5}{2} \text{I} - \frac{5}{2} \text{Y} - 3 \text{X} + \frac{3}{2} \text{Y} + \text{Y} \right) \\ &= \text{III} + \frac{1}{12} \left(4 \text{II} + 5 \text{I} - 4 \text{Y} - 6 \text{X} \right) // \end{aligned}$$

comultiplication

Let $\hat{\Delta}: A(\phi, C) \rightarrow A(\phi, C) \otimes A(\phi, C)$ be the map defined by

$$\hat{\Delta}(D) = \sum_{D_1 \sqcup D_2 = D} D_1 \otimes D_2.$$

Rem

$$\alpha \in A(\phi, C)$$

$$\hat{\Delta}(\alpha) = \alpha \otimes \phi + \phi \otimes \alpha \text{ (i.e. } \alpha \text{ is primitive)}$$

$\Leftrightarrow \alpha$ is a linear combination of connected Jacobi diagrams.

Lemma ([0] Lemma 6.10)

$$\alpha \in A(\phi, C).$$

$$\hat{\Delta}(\alpha) = \alpha \otimes \alpha \text{ (i.e. } \alpha \text{ is grouplike)}$$

$\Leftrightarrow \exists \beta \in A(\phi, C)$: linear combination of conn. Jacobi diagrams (primitive)
s.t. $\alpha = \exp \beta$.

$\langle, \rangle_s$

D_1, D_2 : Jacobi diagrams based on $(X, C \cup S)$,

$S = \{s_1, \dots, s_n\}$, Let us define

$$\left\langle \begin{array}{c} \boxed{D_1} \\ \begin{array}{c} \text{--- } s_1 \\ \vdots \\ \text{--- } s_1 \\ \vdots \\ \text{--- } s_n \\ \vdots \\ \text{--- } s_n \end{array} \end{array} \right\}_{k_1} \dots \left\}_{k_n} \quad \ell_1 \left\{ \begin{array}{c} \text{--- } s_1 \\ \vdots \\ \text{--- } s_1 \\ \vdots \\ \text{--- } s_n \\ \vdots \\ \text{--- } s_n \end{array} \right. \boxed{D_2} \right\}_{\ell_n} \right\rangle_s$$

$$= \begin{cases} \begin{array}{c} \boxed{D_1} \quad \begin{array}{c} \boxed{S_{k_1}} \\ \vdots \\ \boxed{S_{k_n}} \end{array} \quad \boxed{D_2} \\ 0 \end{array} & \begin{array}{l} \text{if } k_i = \ell_i \text{ for all } i \\ \text{otherwise.} \end{array} \end{cases}$$

For a Jacobi diagram D , we denote $\exp D = [D]$.

Example

$$\left[\begin{array}{c} c' \\ \vdots \\ c \end{array} \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\begin{array}{c} c' \quad \dots \quad c' \\ \vdots \quad \dots \quad \vdots \\ c \quad \dots \quad c \end{array}}_{n \text{ times}}, \quad \left[\begin{array}{c} c \\ \vdots \\ c \end{array} \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\begin{array}{c} c \quad \dots \quad c \\ \vdots \quad \dots \quad \vdots \\ c \quad \dots \quad c \end{array}}_{n \text{ times}}.$$

Rem

For $\alpha \in \mathcal{A}(X, \text{cut}\{s\})$, and $\chi_s: \mathcal{A}(X, \text{cut}\{s\}) \rightarrow \mathcal{A}(X \uparrow^s, C)$,

$$\langle \underbrace{\begin{bmatrix} s \\ i \end{bmatrix}}_s, \alpha \rangle_s = \chi_s(\alpha).$$

Let $C = \{c_1, c_2, \dots, c_n\}$.

For a rational symmetric $n \times n$ matrix,

$$\text{we denote } [L] = \left[\sum_{i,j=1}^n L_{ij} \begin{array}{c} c_j \\ \vdots \\ c_i \end{array} \right] \in \mathcal{A}(\emptyset, C).$$

S-substantial

A Jacobi diagram D based on $(X, \text{cut}\{s\})$ is S -substantial

if it has no strut component both ends are labeled by S .

$$X \quad D = \begin{array}{c} s_2 \\ \vdots \\ s_1 \end{array} \begin{array}{c} c_3 \\ \diagup \quad \diagdown \\ c_1 \quad c_2 \end{array} \quad \bigcirc \begin{array}{c} c_2 \\ \vdots \\ c_1 \end{array} \begin{array}{c} s_3 \\ \diagup \quad \diagdown \\ s_1 \quad s_2 \end{array}$$

Gaussian

$G \in \mathcal{A}(X, \text{cut}\{s\})$ is Gaussian w.r.t. S

$$\Leftrightarrow G = \left[\frac{L}{2} \right] \amalg P, \quad \text{where}$$

$P \in \mathcal{A}(X, \text{cut}\{s\})$: S -substantial,

L : rational symmetric $S \times S$ matrix

formal Gaussian integral

$$G = \left[\frac{L}{2} \right] \amalg P : \text{Gaussian w.r.t. } S, \text{ and } \det L \neq 0.$$

Then, we define

$$\int_S G := \langle \left[-\frac{L^{-1}}{2} \right], P \rangle_S \in \mathcal{A}(X, C).$$

Example

$$L = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}, \quad P = \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ 2 \quad 1 \end{array} + \begin{array}{c} 1 \\ \diagdown \quad \diagup \\ 2 \end{array}^c \in \mathcal{A}(\emptyset, \text{cut}\{1,2\})$$

$$(L^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix})$$

$$G = \left[\frac{L}{2} \right] \amalg P.$$

$$\int_{\{1,2\}} G = \langle \left[-\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right], \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ 2 \quad 1 \end{array} + \begin{array}{c} 1 \\ \diagdown \quad \diagup \\ 2 \end{array}^c \rangle_{\{1,2\}} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3}.$$

Category of 2-tangles in $[-1,1]^3$

$\mathcal{T}_2$: the (nonstrict monoidal) category of 2-tangles in $[-1,1]^3$.

object: the free non-associative magma gen. by $\{+, -\}$,

morphism: framed oriented tangles $\gamma \in \mathcal{T}_2(u, v)$

$$(1) \# \partial \gamma = |u| + |v|,$$

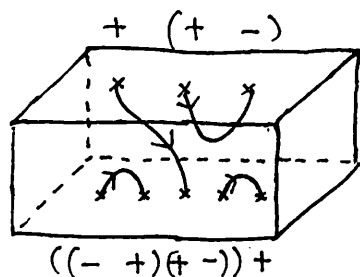
$$(2) \partial \gamma \text{ are uniformly distributed along } [-1,1] \times \{0\} \times \{\pm 1\},$$

(3) In the top square,

γ starts from "+" and ends at "-".

In the bottom square,

γ starts from "-" and ends at "-".



composition & tensor product

$$\gamma_1 \circ \gamma_2 = \begin{array}{|c|} \hline \gamma_2 \\ \hline \gamma_1 \\ \hline \end{array}, \quad \gamma_1 \otimes \gamma_2 = \begin{array}{|c|c|} \hline \gamma_1 & \gamma_2 \\ \hline \end{array}.$$

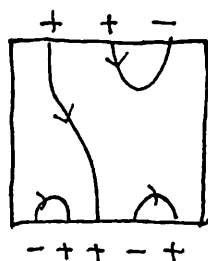
Category of Jacobi diagrams on 1-mfds

(oriented)

$\mathcal{A}$: the (strict monoidal) category of Jacobi diagrams on 1-mfds.

object: associative words in the letter $\{+, -\}$.

morphism: $\mathcal{A}(u, v) = \bigcup_{\text{corr. to } (u, v)} \mathcal{A}(X)$



Thm [LM]

There exists a tensor-preserving functor

$$\hat{\mathbb{Z}} : \mathcal{T}_g \longrightarrow \mathcal{A}$$

$$\gamma \longmapsto \hat{\mathbb{Z}}_\gamma(\gamma) \in \mathcal{A}(\gamma, \emptyset).$$

Values of $\hat{\mathbb{Z}}$

$$\hat{\mathbb{Z}} \left(\begin{array}{c} + \quad (+ +) \\ \downarrow \quad \swarrow \quad \downarrow \\ (+ +) \quad + \end{array} \right) = \Phi, \quad \hat{\mathbb{Z}} \left(\begin{array}{c} (+ +) \quad + \\ \downarrow \quad \swarrow \quad \downarrow \\ + \quad (+ +) \end{array} \right) = \Phi^{-1},$$

$$\hat{\mathbb{Z}} \left(\begin{array}{c} (+ +) \\ \swarrow \quad \searrow \\ (+ +) \end{array} \right) = \left[\frac{1}{2} \right] \text{ (crossing) }, \quad \hat{\mathbb{Z}} \left(\begin{array}{c} (+ +) \\ \swarrow \quad \searrow \\ (+ +) \end{array} \right) = \left[-\frac{1}{2} \right] \text{ (crossing) },$$

$$\hat{\mathbb{Z}} \left(\begin{array}{c} (+ -) \\ \swarrow \quad \searrow \\ (+ -) \end{array} \right) = \nu \frac{1}{2} \text{ (cup) }, \quad \hat{\mathbb{Z}} \left(\begin{array}{c} (+ -) \\ \swarrow \quad \searrow \\ (+ -) \end{array} \right) = \nu \frac{1}{2} \text{ (cap) }.$$

Since

$$\hat{\mathbb{Z}} \left(\begin{array}{c} + \\ \downarrow \quad \downarrow \\ + \end{array} \right) = \begin{array}{|c|c|} \hline \hat{\mathbb{Z}}(\downarrow) & \hat{\mathbb{Z}}(\downarrow) \\ \hline \hat{\mathbb{Z}} \left(\begin{array}{c} + \quad (- +) \\ \downarrow \quad \swarrow \quad \downarrow \\ (+ -) \quad + \end{array} \right) & \\ \hline \hat{\mathbb{Z}}(\downarrow) & \hat{\mathbb{Z}}(\downarrow) \\ \hline \end{array} = \begin{array}{c} \nu \frac{1}{2} \\ \boxed{S_2 \Phi} \\ \nu \frac{1}{2} \end{array} = \begin{array}{c} \downarrow \\ \boxed{S_2 \Phi} \\ \downarrow \end{array},$$

$$\hat{\mathbb{Z}} \left(\begin{array}{c} + \\ \downarrow \\ + \end{array} \right)$$

"

$\downarrow$

$$\text{we have } \nu = \left(\frac{\boxed{S_2 \Phi}}{\boxed{S_2 \Phi}} \right)^{-1} \in \mathcal{A}(\downarrow).$$

If we choose a rational and even Drinfeld associator,

$$\Phi = 1 + \frac{1}{24} \text{ (diagram) } + (\deg \geq 4) \in \mathcal{A}(\downarrow \downarrow), \quad \nu = 1 + \frac{1}{48} \text{ (diagram) } + (\deg \geq 4) \in \mathcal{A}(\downarrow),$$

We modify $\hat{\mathbb{Z}}$, and define $\mathbb{Z} : \mathcal{T}_g \rightarrow \mathcal{A}$ by

$$\mathbb{Z} \left(\begin{array}{c} (+ -) \\ \swarrow \quad \searrow \\ (+ -) \end{array} \right) = \text{cup}, \quad \mathbb{Z} \left(\begin{array}{c} (+ -) \\ \swarrow \quad \searrow \\ (+ -) \end{array} \right) = \text{cap} \otimes \nu,$$

and for other elementary g -tangles, $\mathbb{Z} = \hat{\mathbb{Z}}$.

$$\Sigma(\text{diagram}) = \text{diagram} + \text{diagram} + \frac{1}{2} \text{diagram} + \frac{1}{48} \text{diagram} + \frac{1}{6} \text{diagram} + \frac{1}{24} (\text{diagram} - \text{diagram}) + \frac{1}{48} \text{diagram} + (\text{deg} \geq 4).$$
$$\mathbb{Z} \left(\begin{array}{c} \text{Diagram of a knot with four strands and crossings} \end{array} \right) = \mathbb{Z} \left(\begin{array}{c} \text{Diagram of a braid with four strands and crossings} \end{array} \right) =$$

$$\begin{aligned}
&= \text{diagram 1} + \frac{1}{24} \text{diagram 2} + \frac{1}{24} \text{diagram 3} - \frac{1}{24} \text{diagram 4} - \frac{1}{24} \text{diagram 5} + \frac{1}{48} \text{diagram 6} + (\deg \geq 4) \\
&= \text{diagram 7} + \text{diagram 8} + \frac{1}{2} \text{diagram 9} + \frac{1}{6} \text{diagram 10} \\
&\quad + \frac{1}{24} \left(\text{diagram 11} + \text{diagram 12} + \text{diagram 13} + \text{diagram 14} - \text{diagram 15} - \text{diagram 16} - \text{diagram 17} - \text{diagram 18} \right) \\
&\quad + \frac{1}{48} \left(\text{diagram 19} + \text{diagram 20} \right) + (\deg \geq 4) \\
&= \text{diagram 21} + \text{diagram 22} + \frac{1}{2} \text{diagram 23} + \frac{1}{48} \text{diagram 24} + \frac{1}{6} \text{diagram 25} \\
&\quad + \frac{1}{24} \left(\text{diagram 26} - \text{diagram 27} - \text{diagram 28} + \text{diagram 29} \right) + \frac{1}{48} \text{diagram 30} + (\deg \geq 4) \\
&= (\text{RHS}). //
\end{aligned}$$

Example

2-4

Assume that chosen Drinfeld associator is rational and even.

$$x^{-1}Z\left(\begin{array}{c} x \\ \nearrow \searrow \\ x \end{array}\right) = \left[\begin{array}{c} 1 \\ x \end{array} \right] \perp \exp_{\perp} \left(\emptyset + \frac{1}{8} \begin{array}{c} \nearrow \\ \searrow \end{array} + \frac{1}{48} \begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} - \frac{1}{8} \begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} + (i - \deg \geq 3) \right).$$

☺ Recall that

$$Z\left(\begin{array}{c} x \\ \nearrow \searrow \\ x \end{array}\right) = \begin{array}{c} \rightarrow \\ \leftarrow \end{array} + \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{2} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{48} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{6} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{24} (\begin{array}{c} \rightarrow \\ \rightarrow \end{array} - \begin{array}{c} \rightarrow \\ \rightarrow \end{array}) + \frac{1}{48} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + (\deg \geq 4).$$

$$\begin{aligned} \therefore x^{-1}_x \left(\begin{array}{c} x \\ \nearrow \searrow \\ x \end{array} \right) &= \begin{array}{c} \rightarrow \\ \leftarrow \end{array} + \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{2} (\begin{array}{c} \rightarrow \\ \rightarrow \end{array} - \frac{1}{2} \begin{array}{c} \rightarrow \\ \rightarrow \end{array}) + \frac{1}{48} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{6} \tau(\begin{array}{c} \rightarrow \\ \rightarrow \end{array}) \\ &+ \frac{1}{24} \{ \begin{array}{c} \rightarrow \\ \rightarrow \end{array} - (\begin{array}{c} \rightarrow \\ \rightarrow \end{array} - \frac{1}{2} \begin{array}{c} \rightarrow \\ \rightarrow \end{array}) \} - \frac{1}{48} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + (\deg \geq 4) \\ &= \begin{array}{c} \rightarrow \\ \leftarrow \end{array} + \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{2} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{8} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{48} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \\ &+ \frac{1}{6} \{ \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{12} (4 \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + 5 \begin{array}{c} \rightarrow \\ \rightarrow \end{array} - 4 \begin{array}{c} \rightarrow \\ \rightarrow \end{array} - 6 \begin{array}{c} \rightarrow \\ \rightarrow \end{array}) \} \\ &+ \frac{1}{48} \{ 3 \begin{array}{c} \rightarrow \\ \rightarrow \end{array} - 2 \begin{array}{c} \rightarrow \\ \rightarrow \end{array} - \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \} + (\deg \geq 4) \\ &= \begin{array}{c} \rightarrow \\ \leftarrow \end{array} + \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{2} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{8} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{48} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{6} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \\ &+ \frac{4}{72} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{5}{72} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{144} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} - \frac{1}{8} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{48} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + (\deg \geq 4) \\ &\quad \frac{1}{16} (\begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \begin{array}{c} \rightarrow \\ \rightarrow \end{array}) \end{aligned}$$

$$\frac{2}{3} \tau \frac{1}{3} \Psi \bar{\Psi}$$

$$\begin{aligned} \therefore x^{-1}_{\{x,y\}} \left(\begin{array}{c} x \\ \nearrow \searrow \\ x \end{array} \right) &= \emptyset + i + \frac{1}{2} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{8} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{48} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{6} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{8} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} - \frac{1}{8} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} - \frac{1}{288} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + (\deg \geq 4). \\ &= [1] \perp \exp_{\perp} \left(\emptyset + \frac{1}{8} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + \frac{1}{48} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} - \frac{1}{8} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} + (\deg \geq 4) \right) \\ &= \text{RHS.} \end{aligned}$$

↑ For connected diagrams, $i - \deg \geq \deg - 1$. //

Homology cubes

A homology cube is a pair (M, m) , where

M is a cpt conn. oriented 3-mfd s.t. $H_*(M) \cong H_*([-1, 1]^3)$.

$m: \partial[-1, 1]^3 \xrightarrow{\cong} \partial M$: ori-pres. homeo.

Category of $\mathcal{B}$ -tangles in homology cubes

$\mathcal{T}_{\mathcal{B}} \text{Cub}$: the (nonstrict monoidal) category of $\mathcal{B}$ -tangles in homology cubes

object : the free non-ass. magma gen. by $\{+, -\}$,

morphism : framed oriented tangles σ in homology cubes B

$$(B, \sigma) \in \mathcal{T}_{\mathcal{B}} \text{Cub}(u, v)$$

$$(1) \# \partial \sigma = |u| + |v|$$

$$(2) \partial \sigma \text{ are uniformly distributed along } m([-1, 1] \times \{0\} \times \{\pm 1\}).$$

$$(3) \text{ The orientation of } \partial \sigma \text{ corr. to } u \text{ and } v.$$

$$\underline{U_{\pm} \in \mathcal{A}(\emptyset)}$$

$$\begin{aligned} U_{\pm} &= \int \chi_{\mathcal{O}}^{-1} \left(\underbrace{\nu \# \mathcal{Z}(\sigma_{\pm 1})}_{\substack{\mathcal{A}(\sigma, \emptyset) \\ \mathcal{A}(\{*\})}} \right) = \int \chi_{\mathcal{O}}^{-1} \left(\left[\begin{array}{c} \pm \frac{1}{2} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right] \right) \\ &= \int \exp \left(\pm \frac{1}{2} i \right) \perp (\phi + \frac{1}{16} \phi + \deg \geq 3) \\ &= \phi \mp \frac{1}{16} \text{---} + \deg \geq 3 \in \mathcal{A}(\emptyset). \end{aligned}$$

Normalization

$L \cup \sigma$: a $\mathcal{B}$ -tangle in $[-1, 1]^3$.

We set $\mathcal{Z}(L \cup \sigma) := \nu^{\otimes \pi_0(L)} \#_{\pi_0(L)} \mathcal{Z}(L \cup \sigma) \in \mathcal{A}(L \cup \sigma)$.

Example

$$L = \bigcirc \uparrow = \sigma \subset [-1, 1]^3$$

$$\begin{aligned} \mathcal{Z}(L \cup \sigma) &= \mathcal{Z}(\bigcirc^{\nu}) \perp \mathcal{Z}(\uparrow_+) \\ &= \text{---} \uparrow \in \mathcal{A}(\bigcirc \uparrow). \end{aligned}$$

Surgery presentation

$(B, \tau) \in \mathcal{T}_q \text{Cub}(u, v)$.

(L, τ) : a surgery presentation of (B, τ) .

$\Leftrightarrow$ (1) L is a framed oriented link in $[-1, 1]^3$ s.t. $[-1, 1]^3_L = B$,
def

(2) τ is a tangle in $[-1, 1]^3$ s.t.

surgery along L maps $([-1, 1]^3, \tau)$ to (B, τ) .

Kontsevich-LMO invariant

$(B, \tau) \in \mathcal{T}_q \text{Cub}(u, v)$.

$$\mathbb{Z}(B, \tau) := U_+^{-\sigma_+(L)} \amalg U_-^{-\sigma_-(L)} \amalg \int_{\pi_0(L)} \chi_{\pi_0(L)}^{-1} \mathbb{Z}(L \cup \tau) \in \mathcal{A}(\tau),$$

where (L, τ) is a surgery presentation of (B, τ) ,

$(\sigma_+(L), \sigma_-(L))$ denotes the signature of the linking matrix of L .

Rem

When $B = [-1, 1]^3$ ($L = \emptyset$),

$\mathbb{Z}(B, \tau) = \mathbb{Z}(\tau) \in \mathcal{A}(\tau)$: (modified) Kontsevich integral.

When $\tau = \emptyset$,

$$\mathbb{Z}(B, \emptyset) = U_+^{-\sigma_+(L)} \amalg U_-^{-\sigma_-(L)} \amalg \int_{\pi_0(L)} \chi_{\pi_0(L)}^{-1} \mathbb{Z}(L) \in \mathcal{A}(\emptyset)$$

: LMO invariant.

Thm

$\mathbb{Z}(B, \tau)$ does not depend on the choice of a surgery presentation.

Lemma 3.17

(B, τ) : a bottom-top q -tangle in a homology cube B .

$\chi^{-1} \mathbb{Z}(B, \tau) \in \mathcal{A}(\emptyset, \pi_0(\tau))$ is grouplike,

and its strut-part is $\left[\frac{Lk_B(\tau)}{2} \right]$.

bottom-top tangle

For $g \geq 1$, take $(2g)$ -points in $[0,1]^2$ as below.

$$[0,1]^2 = \begin{array}{|c|} \hline \begin{array}{ccccccc} x & x & x & x & \cdots & x & x \\ p_1 & q_1 & p_2 & q_2 & & p_g & q_g \end{array} \\ \hline \end{array}$$

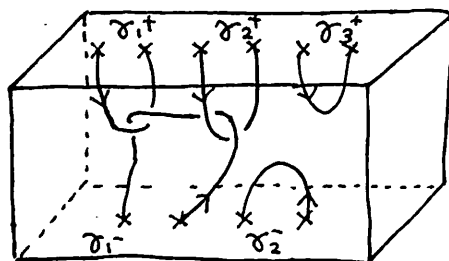
(B, γ) is a bottom-top tangle of type (g_+, g_-)

$\Leftrightarrow B = (B, b)$ is a cobordism from $[0,1]^2$ to $[0,1]^2$, and

$\gamma = (\gamma^+, \gamma^-)$ is a framed oriented tangle s.t.

γ_j^+ runs from $p_j \times \{1\}$ to $q_j \times \{1\}$.

γ_j^- runs from $q_j \times \{-1\}$ to $p_j \times \{-1\}$.



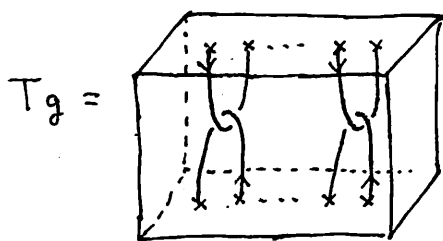
type $(3, 2)$

composition & tensor product

(B, γ) : bottom-top tangle of type (g_+, g_-) ,

(C, ν) :

(h_+, h_-) , $g_+ = h_-$.



$T_g =$

$\subset [0,1]^3$.

$(B, \gamma) \circ (C, \nu) =$

$\nu \subset C$
$T_{g_+} \subset [-1,1]^3$
$\gamma \subset B$

& surgery along $\gamma^+ \cup T_{g_+} \cup \nu^-$.

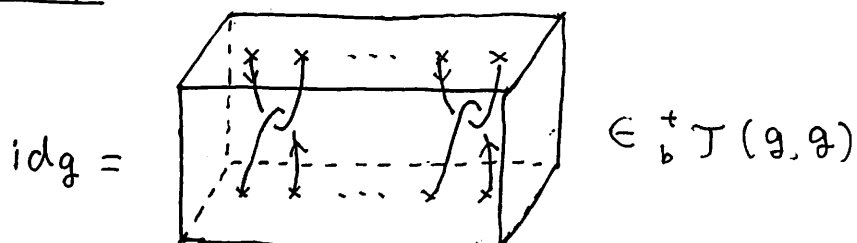
Category of bottom-top tangles ${}^t_b \mathcal{T}$

${}^t_b \mathcal{T}$: the (strict monoidal) category of bottom-top tangles

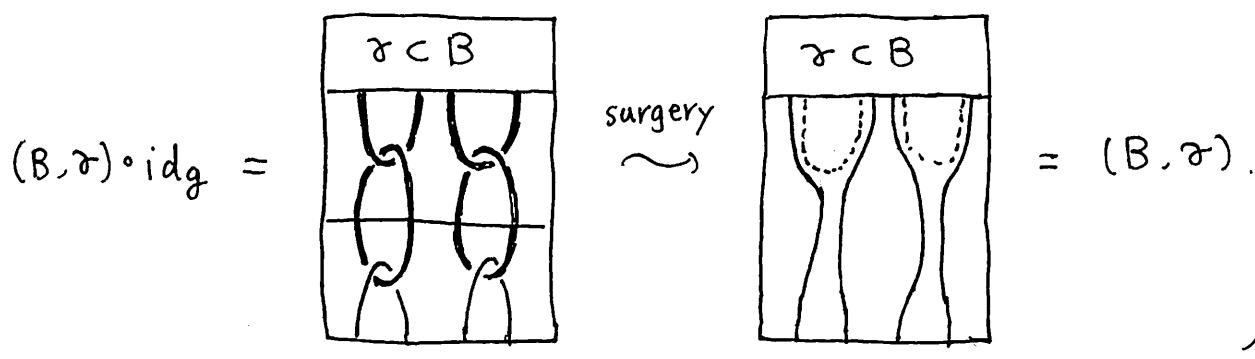
object : $\mathbb{Z}_{\geq 0}$,

morphism : $(B, b) \in {}^t_b \mathcal{T}(g_+, g_-)$: bottom-top tangle of type (g_+, g_-) .

identity



☹



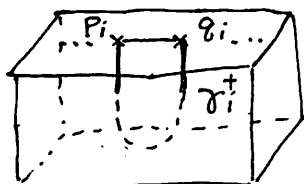
linking matrix

(B, γ) : a bottom-top tangle in a homology cube,

$$\hat{B} := B \cup_b (S^3 \setminus [-1, 1]^3).$$

$$\hat{\gamma}_i^+ = \gamma_i^+ \cup \overline{p_i q_i}$$

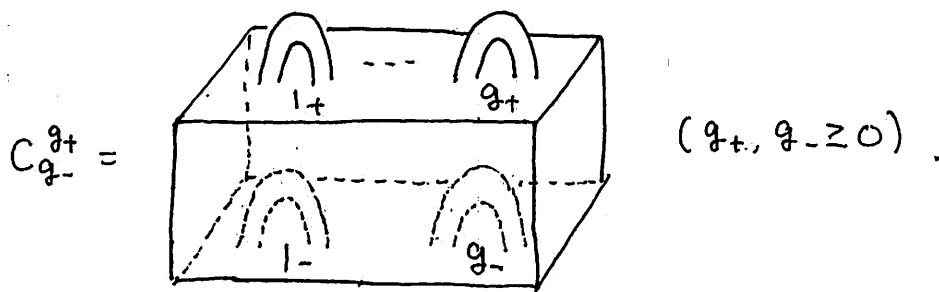
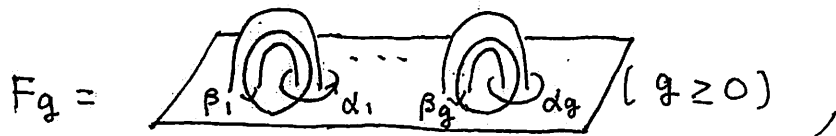
$$\hat{\gamma}_i^- = \gamma_i^- \cup \overline{p_i q_i}$$



Let us define the linking matrix of γ by

$$Lk_B(\gamma) := Lk_{\hat{B}}(\hat{\gamma}).$$

cobordism



(M, m) : a cobordism from F_{g_+} to F_{g_-}

$\Leftrightarrow M$: cpt conn. ori. 3-mfd ,

$m : \partial C_{g_-}^{g_+} \xrightarrow{\cong} \partial M$ ori.-pres. homeo .

Category of cobordisms Cob

Cob : the (strict monoidal) category of cobordisms

object : $\mathbb{Z}_{\geq 0}$

morphism : $(M, m) \in \text{Cob}(g_+, g_-)$: a cobordism from F_{g_+} to F_{g_-}

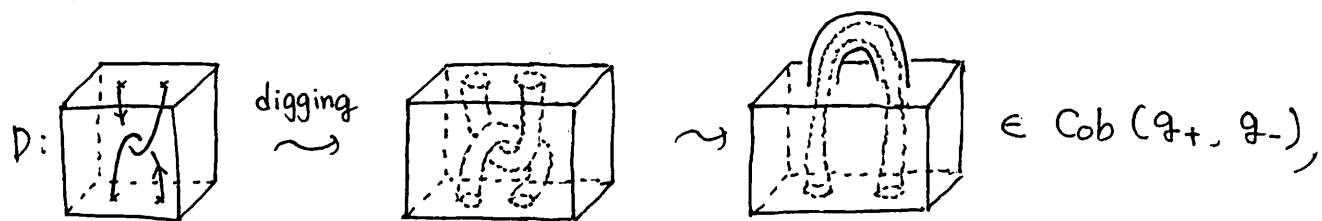
$$M \circ N = \begin{array}{|c|} \hline M \\ \hline N \\ \hline \end{array} , \quad M \otimes N = \begin{array}{|c|c|} \hline M & N \\ \hline \end{array} .$$

Thm 2.10

$\exists D : {}^t_b \mathcal{T} \rightarrow \text{Cob} : \text{isomorphism.}$

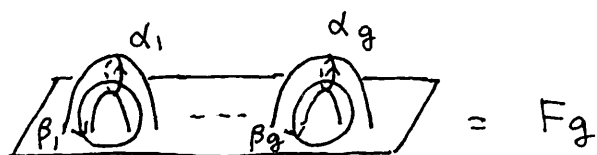
Proof

$$(B, \tau) \in {}^t_b \mathcal{T}(g_+, g_-)$$



where the markings of $F_{g_{\pm}}$ are given by the meridians and longitudes of tangles.

$$(M, m) \in \text{Cob.}$$



By attaching 2-hdles along $m_-(\alpha_i)$ and $m_+(\beta_j)$ ($1 \leq i \leq g_-$, $1 \leq j \leq g_+$).

we obtain a cobordism from $[0,1]^2$ to $[0,1]^2$,

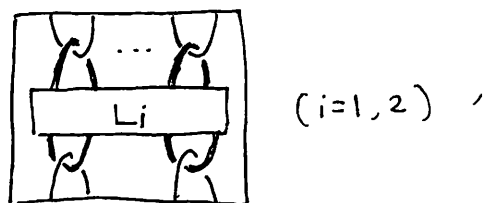
and a tangle comes from the cocores of these handles.

This construction gives the inverse of D.

• functoriality

$(B, \tau) \in {}^t_b \mathcal{T}(g_+, g_-)$ and $(C, \nu) \in {}^t_b \mathcal{T}(h_+, h_-)$ can be written as

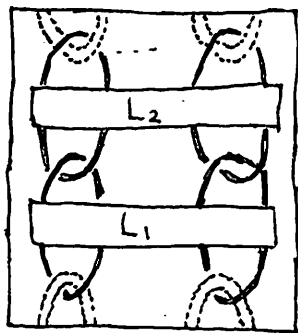
the form



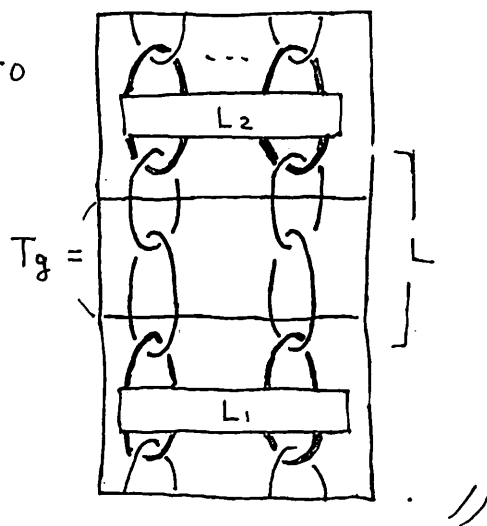
where thick lines and L_i are links along which surgeries applied.

Then, the composition of corr. 3-mfds M, N is

$$M \circ N =$$



and it corresponds to



& surgery along L .

Lagrangian cobordism

$$A_g = \langle \alpha_1, \alpha_2, \dots, \alpha_g \rangle \subset H_1(F_g),$$

$$B_g = \langle \beta_1, \beta_2, \dots, \beta_g \rangle.$$

$(M, m) \in \text{Cob}(g_+, g_-)$ is Lagrangian

$\Leftrightarrow$ (1) $m_- \oplus m_+ : A_g \oplus B_{g_+} \rightarrow H_1(M)$ is an isomorphism,

(2) $m_+(A_{g_+}) \subset m_-(A_g)$ in $H_1(M)$.

Rem

$$(1) \quad \iota : \text{MCG}(F_g) \rightarrow \text{Cob}(g, g)$$

$$h \mapsto (F_g \times [-1, 1], (\text{Id} \times \{-1\}) \cup (h \times \{1\}))$$

is an injective homo.

$$(2) \quad \iota(h) \text{ is Lagrangian} \Leftrightarrow h(A_g) = A_g.$$

Lemma 2.12

$$(B, \sigma) \in {}^+_6\mathcal{T}(g_+, g_-).$$

$D(B, \sigma)$ is Lagrangian $\Leftrightarrow \text{Lk}(\sigma^+)$ is trivial, and B is a homology cube.

Proof

$$(M, \mathcal{M}) := D(B, \sigma).$$

Since $B = M \cup \left(\bigcup_{i=1}^{g_+} N(\sigma_i^+) \cup \bigcup_{i=1}^{g_-} N(\sigma_i^-) \right)$, we have

$$\begin{aligned} H_2(B) &\rightarrow H_1\left(\bigcup_{i=1}^{g_+} \partial N(\sigma_i^+) \cup \bigcup_{i=1}^{g_-} \partial N(\sigma_i^-)\right) \\ &\rightarrow H_1(M) \oplus \underbrace{H_1\left(\bigcup_{i=1}^{g_+} N(\sigma_i^+) \cup \bigcup_{i=1}^{g_-} N(\sigma_i^-)\right)}_{=0} \rightarrow H_1(B). \end{aligned}$$

$$\therefore H_1(B) = H_2(B) = 0.$$

$$\Leftrightarrow H_1\left(\bigcup_{i=1}^{g_+} \partial N(\sigma_i^+) \cup \bigcup_{i=1}^{g_-} \partial N(\sigma_i^-)\right) \cong H_1(M)$$

(i.e. $H_1(M)$ is freely generated by the meridians of the tangle $\sigma_i^\pm$)

$$\Leftrightarrow H_1(M) = \langle \alpha_1^-, \dots, \alpha_{g_-}^-, \beta_1^+, \dots, \beta_{g_+}^+ \rangle$$

$$\Leftrightarrow H_1(M) = \mathcal{M}_-(A_{g_-}) \oplus \mathcal{M}_+(B_{g_+}), \dots (*)$$

On the other hand,

$\text{Lk}(\sigma^+)$ is trivial

$$\Leftrightarrow \text{Lk}(\alpha_i^+, \alpha_j^+) = 0 \quad (1 \leq i \leq j \leq g_+).$$

When (*) is satisfied, $\text{Lk}(x, \alpha_1^+) = \dots = \text{Lk}(x, \alpha_{g_+}^+) = 0 \Leftrightarrow x \in \mathcal{M}_-(A_{g_-})$.

Thus, $\text{Lk}(\alpha_i^+, \alpha_j^+) = 0 \Leftrightarrow \mathcal{M}_+(A_{g_+}) \subset \mathcal{M}_-(A_{g_-})$. //

Lagrangian g -cobordism

$(M, m, w_t(M), w_b(M))$ is a Lagrangian g -cobordism

$\Leftrightarrow (M, m)$ is a Lagrangian cobordism from F_g to F_f , and
 $w_t(M)$ and $w_b(M)$ are non-ass. words of length g and f
 in the single letter $\bullet$.

Category of Lagrangian g -cobordisms $\mathcal{L}Cob_g$

$\mathcal{L}Cob_g$: the (non-strict monoidal) category of Lagrangian g -cobordisms

object: the free non-ass. magma gen. by $\bullet$,

morphism: $(M, m, w_t(M), w_b(M)) \in \mathcal{L}Cob(u, v)$,

where (M, m) is a Lagrangian cobordism from $|u|$ to $|v|$.

Rem

$D^{-1}: Cob \xrightarrow{\cong} {}_b^t \mathcal{T}$ induces a functor $\mathcal{L}Cob_g \rightarrow \mathcal{T}_g Cub$.

$$\begin{aligned} \bullet &\mapsto (+-) \\ (M, m) &\mapsto D^{-1}(M, m) \end{aligned}$$

$\mathcal{Z}(M)$

For a Lagrangian g -cobordism (M, m) from F_g to F_f ,

We denote $\mathcal{Z}(M) = \mathcal{Z}(B, \sigma) \in \mathcal{A}(\sigma) = \mathcal{A}(\hookrightarrow^{Lg} \hookleftarrow^{Lf}, \emptyset)$

where $Lg = \{1, 2, \dots, g\}$.

Rem

By Lemma 2.12 and Lemma 3.17, $\chi^{-1} \mathcal{Z}(M) \in \mathcal{A}(\emptyset, Lg^+ \cup Lf^-)$

$$\begin{array}{ll} \text{Lk}(\sigma^+) & \text{strut-part} \\ \text{is trivial} & \text{is } \left[\frac{\text{Lk}_B(\sigma)}{2} \right] \end{array}$$

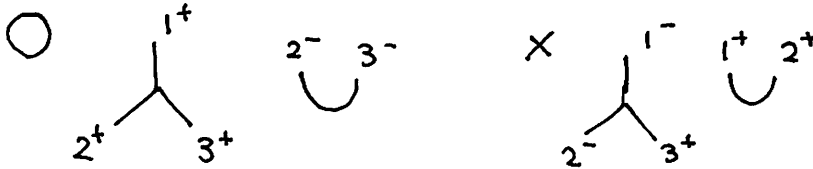
is Lg^+ -substantial.

§ LMO functor $\tilde{Z} : \mathcal{L}Cob_g \rightarrow {}^{ts}\mathcal{A}$

top-substantial

$$f, g \geq 0.$$

A Jacobi diagram based on $\mathcal{A}(\emptyset, [g]^+ \cup [f]^-)$ is top-substantial if it is LgT^+ -substantial.



${}^{ts}\mathcal{A}$

The (strict monoidal) category of top-substantial diagrams.

object: $\mathbb{Z}_{\geq 0}$,

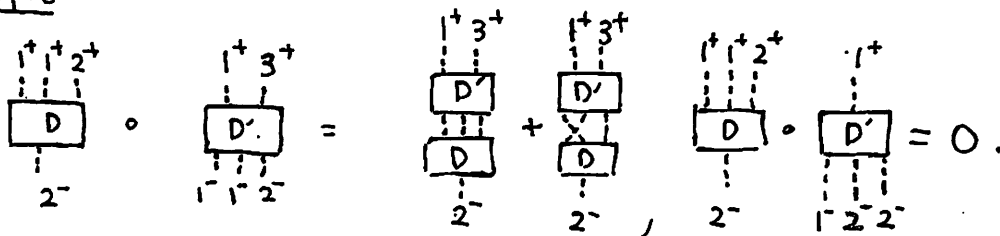
morphism: ${}^{ts}\mathcal{A}(g, f)$: the subspace of $\mathcal{A}(\emptyset, [g]^+ \cup [f]^-)$ spanned by top-substantial diagrams.

composition & tensor product

$$x \in {}^{ts}\mathcal{A}(g, f), y \in {}^{ts}\mathcal{A}(h, g)$$

$$\begin{aligned} x \circ y &= \langle (x/i^+ \mapsto i^*), (y/i^- \mapsto i^*) \rangle_{LgT^*} \\ &= \left(\begin{array}{l} \text{sum of all ways of gluing the } i^+ \text{-colored vertices of } x \\ \text{to the } i^- \text{-colored vertices of } y \text{ for all } i=1, \dots, g \end{array} \right). \end{aligned}$$

Example



identity

$$id_g = \left[\begin{array}{c} 1^+ \\ \vdots \\ g^+ \end{array} \right] + \dots + \left[\begin{array}{c} g^+ \\ \vdots \\ 1^+ \end{array} \right].$$

Rem

For a Lagrangian g -cobordism (M, m) from F_g to F_f ,

$\chi^{-1} \tilde{Z}(M) \in \mathcal{A}(\emptyset, LgT^+ \cup LfT^-)$ is in ${}^{ts}\mathcal{A}(g, f)$.

Lemma 4.5

$a \in {}^{ts}A(g, f)$, $b \in {}^{ts}A(h, g)$ s.t.

$$a = \left[\frac{A}{2} \right] \perp a^Y \text{ and } b = \left[\frac{B}{2} \right] \perp b^Y,$$

where A is a symmetric $(Lg\gamma^+ \cup Lf\gamma^-) \times (Lg\gamma^+ \cup Lf\gamma^-)$ matrix
and B $(Lh\gamma^+ \cup Lg\gamma^-) \times (Lh\gamma^+ \cup Lg\gamma^-)$ matrix
of the form

$$A = \begin{pmatrix} 0 & A^{+-} \\ A^{-+} & A^{--} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & B^{+-} \\ B^{-+} & B^{--} \end{pmatrix},$$

$$\text{Then, } a \circ b = \left[\frac{1}{2} \begin{pmatrix} 0 & B^{+-} A^{+-} \\ A^{-+} B^{+-} & A^{--} + A^{-+} B^{--} A^{+-} \end{pmatrix} \right] \perp (a^Y \star^{A,B} b^Y),$$

where

$$x \star^{A,B} y = \langle (x/i^+ \mapsto i^* + B^{+-} i^- + A^{-+} B^{--} i^-), \\ ([B^{--}/2] / i^- \mapsto i^*) \perp (y/i^- \mapsto i^* + A^{-+} i^+) \rangle_{Lg\gamma^*}$$

Rem

When $f = g = h$ and $A = B = \begin{pmatrix} 0 & I_g^{+-} \\ I_g^{-+} & 0 \end{pmatrix}$,

$$x \star y = \langle (x/i^+ \mapsto i^* + i^+), (y/i^- \mapsto i^* + i^-) \rangle_{Lg\gamma^*}.$$

Rem

If a and b is grouplike, $a \circ b$ is grouplike as well.

$$\lambda(x, y; r)$$

$$\begin{aligned} \lambda(x, y; r) &= \chi_r^{-1} \left(\underbrace{\begin{array}{c} x \quad y \\ \parallel \quad \parallel \\ \hline \end{array}}_r \right) \in \mathcal{A}(\emptyset, \{x, y, r\}) \\ &= \chi_r^{-1} \left(\begin{array}{c} \text{---} \\ \parallel \\ r \end{array} + \begin{array}{c} x \\ \parallel \\ \text{---} \\ \parallel \\ r \end{array} + \begin{array}{c} y \\ \parallel \\ \text{---} \\ \parallel \\ r \end{array} + \frac{1}{2} \begin{array}{c} x \quad x \\ \parallel \quad \parallel \\ \hline \end{array} + \frac{1}{2} \begin{array}{c} y \quad y \\ \parallel \quad \parallel \\ \hline \end{array} + \begin{array}{c} x \quad y \\ \parallel \quad \parallel \\ \hline \end{array} + \dots \right) \\ &= \emptyset + \begin{array}{c} x \\ \parallel \\ r \end{array} + \begin{array}{c} y \\ \parallel \\ r \end{array} + \frac{1}{2} \begin{array}{c} x \quad x \\ \parallel \quad \parallel \\ \hline \end{array} + \frac{1}{2} \begin{array}{c} y \quad y \\ \parallel \quad \parallel \\ \hline \end{array} + \begin{array}{c} x \quad y \\ \parallel \quad \parallel \\ \hline \end{array} + \frac{1}{2} \begin{array}{c} x \quad y \\ \diagup \quad \diagdown \\ \parallel \\ r \end{array} + \dots \end{aligned}$$

Lemma

$$\lambda(x, y; r) = \left[\begin{array}{c} x \\ \parallel \\ r \end{array} + \begin{array}{c} y \\ \parallel \\ r \end{array} + \frac{1}{2} \begin{array}{c} x \quad y \\ \diagup \quad \diagdown \\ \parallel \\ r \end{array} + \frac{1}{12} \left(\begin{array}{c} x \quad x \quad y \\ \diagup \quad \diagdown \quad \diagdown \\ \parallel \\ r \end{array} + \begin{array}{c} y \quad y \quad x \\ \diagup \quad \diagdown \quad \diagdown \\ \parallel \\ r \end{array} \right) + \dots \right]:$$

Baker-Campbell-Hausdorff series.

Thus, $\lambda(x, y; r)$ is grouplike.

Proof

For a connected tree with one r -leg $\begin{array}{c} \square \\ \parallel \\ r \end{array} \in \mathcal{A}(\emptyset, \{x, y, r\})$,

$$\begin{aligned} \chi_r(\exp_{\perp}(\begin{array}{c} \square \\ \parallel \\ r \end{array})) &= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\begin{array}{c} \square \dots \square \\ \parallel \dots \parallel \\ \hline \end{array}}_{n \text{ times}} \\ &= \underline{[\begin{array}{c} \square \\ \parallel \\ r \end{array}]}, \end{aligned}$$

$$\therefore \exp_{\perp}(\begin{array}{c} \square \\ \parallel \\ r \end{array}) = \chi_r^{-1}(\underline{[\begin{array}{c} \square \\ \parallel \\ r \end{array}]}) = \chi_r^{-1}(\exp_{\perp}(\begin{array}{c} \square \\ \parallel \\ r \end{array}))$$

The element $\log e^x e^y$ is a linear combination of conn. diagrams,

$$\text{Thus, } \chi_r^{-1} \left(\underbrace{\begin{array}{c} x \quad y \\ \parallel \quad \parallel \\ \hline \end{array}}_r \right) = \chi_r^{-1} \left(\exp_{\perp} \log e^x e^y \right) = \exp_{\perp} \left(\log e^x e^y \right) //$$

$$\pi(x_+, x_-)$$

$$\pi(x_+, x_-) = U_+^{-1} \perp U_-^{-1} \perp \int_{r_{\pm}} \langle \lambda(x_-, y_-; r_-) \perp \lambda(x_+, y_+; r_+), x^{-1} \mathbb{Z}(T_1^y) \rangle_{y_{\pm}} \quad (*)$$

$$(*) = x_{r_{\pm}}^{-1} \left\langle \begin{array}{c} x_+ \quad y_+ \\ \boxed{\quad} \quad \boxed{\quad} \\ \hline r_+ \end{array} \perp \begin{array}{c} x_- \quad y_- \\ \boxed{\quad} \quad \boxed{\quad} \\ \hline r_- \end{array}, x_{y_{\pm}}^{-1} \mathbb{Z} \left(\begin{array}{c} y_+ \\ \curvearrowright \\ y_- \end{array} \right) \right\rangle_{y_{\pm}}$$

$$= x_{r_{\pm}}^{-1} \left\langle \begin{array}{c} x_+ \quad y_+ \\ \boxed{\quad} \quad \boxed{\quad} \\ \hline r_+ \end{array} \perp \begin{array}{c} x_- \quad y_- \\ \boxed{\quad} \quad \boxed{\quad} \\ \hline r_- \end{array}, \begin{array}{c} y_+ \quad y_+ \dots y_+ \\ \boxed{x^{-1} \mathbb{Z}(T_1^y)} \\ y_- \quad y_- \dots y_- \end{array} \right\rangle_{y_{\pm}}$$

$$= x_{r_{\pm}}^{-1} \left(\begin{array}{c} x_+ \\ \boxed{\quad} \xrightarrow{\quad} r_+ \\ \vdots \\ \boxed{x^{-1} \mathbb{Z}(T_1^y)} \\ \vdots \\ \boxed{\quad} \xrightarrow{\quad} r_- \\ x_- \end{array} \right)$$

$$= \begin{array}{c} x_+ \quad r_+ \\ \vdots \quad \vdots \\ \boxed{\quad} \\ \vdots \quad \vdots \\ x_- \quad r_- \end{array} \in \mathcal{A}(\emptyset, \{x_{\pm}, r_{\pm}\}).$$

Lemma 4.9

$\pi(x_+, x_-)$ is grouplike in $\mathcal{A}(\emptyset, \{x_+, x_-\})$, and its strut part is $\begin{bmatrix} x_+ \\ x_- \end{bmatrix}$.

Rem

By straightforward computations, we have

$$\pi(x_+, x_-) = \begin{bmatrix} x_+ \\ x_- \end{bmatrix} \perp \left(\phi - \frac{1}{8} \begin{array}{c} x_+ \\ \bigcirc \\ x_- \end{array} - \frac{1}{48} \begin{array}{c} x_+ \quad x_+ \\ \curvearrowright \\ x_- \end{array} + \frac{1}{8} \begin{array}{c} x_+ \quad x_+ \\ \vdots \quad \vdots \\ x_- \quad x_- \end{array} + (\deg > 3) \right).$$

Proof of Lemma 4.9

• strut part

The strut part of $\chi^+ \Xi(T, \nu)$ coincides with the linking matrix $\begin{bmatrix} - & y_+ \\ & y_- \end{bmatrix}$,
 $\lambda(x, y; r)$ is, by definition, $\begin{bmatrix} x \\ r \end{bmatrix} + \begin{bmatrix} y \\ r \end{bmatrix}$.

($\chi^+ \Xi(T, \nu)$ and $\lambda(x, y; r)$ are grouplike)

$$(*) = \left\langle \begin{bmatrix} x_- & y_- & x_+ & y_+ \\ r_- & r_- & r_+ & r_+ \end{bmatrix} \perp (Y\text{-part}), \begin{bmatrix} - & y_+ \\ & y_- \end{bmatrix} \perp (Y\text{-part}) \right\rangle_{y_{\pm}}$$

$$= \left\langle \begin{bmatrix} x_- & x_+ \\ r_- & r_+ \end{bmatrix} \perp \begin{bmatrix} - & y_+ \\ & y_- \end{bmatrix} \perp (Y\text{-part}) \right\rangle_{y_{\pm}}$$

$$= \left\langle \begin{bmatrix} x_- & x_+ \\ r_- & r_+ \end{bmatrix} \perp (Y\text{-part}) \right\rangle_{y_{\pm}}$$

$$\begin{aligned} \therefore \int_{r_{\pm}} (*) &= \left\langle \begin{bmatrix} r_+ \\ r_- \end{bmatrix}, \begin{bmatrix} x_- & x_+ \\ r_- & r_+ \end{bmatrix} \perp (Y\text{-part}) \right\rangle_{r_{\pm}} \\ &= \left\langle \begin{bmatrix} x_+ \\ x_- \end{bmatrix} \perp (Y\text{-part}) \right\rangle_{r_{\pm}} // \end{aligned}$$

• grouplike

Since χ preserves coproducts, $\chi^+ \Xi(T, \nu)$ is grouplike,
 and $\lambda(x, y; r)$ is also grouplike by definition.

Thm 3.6 (JMM)

$D, E \in \mathcal{A}(\emptyset, C \cup S)$: grouplike.

If D or E is S -substantial, $\langle E, D \rangle_S$ is grouplike.

Since $\lambda(x_+, y_+; r_+) \perp \lambda(x_-, y_-; r_-)$ is $\{y_{\pm}\}$ -substantial,

(*) is also grouplike by Thm 3.6.

$\int_{r_{\pm}} (*) = \left\langle \begin{bmatrix} r_+ \\ r_- \end{bmatrix}, \begin{bmatrix} x_- & x_+ \\ r_- & r_+ \end{bmatrix} \perp (Y\text{-part}) \right\rangle_{r_{\pm}}$ is also grouplike
 by Thm 3.6. //

$$\pi_g = \pi(1^+, 1^-) \perp \dots \perp \pi(g^+, g^-)$$

$$= v_+^{-g} \perp v_-^{-g} \perp \int_r < \bigoplus_{i=1}^g \lambda(x_i^-, y_i^-, r_i^-) \perp \bigoplus_{i=1}^g \lambda(x_i^+, y_i^+, r_i^+), \chi^{-1} z(T_g^v) >_y \\ \in A(\emptyset, LgT^+ \cup LgT^-)$$

Rem

$\overline{\pi}_g$ is a grouplike element, and the strut part is $\text{Id}_g = \begin{bmatrix} \sum_{i=1}^g & \\ & 1^+ \\ & & - \end{bmatrix}$.

Lemma 4.10

$$M \in \mathcal{L} \text{Cob}_g(W_1, W_2),$$

$$N \in \mathcal{L} \text{Cob}_g(w_2, w_3)$$

Then, $\chi^{-1} \mathbb{Z}(M \cdot N) = \chi^{-1} \mathbb{Z}(M) \cdot \pi_g \cdot \chi^{-1} \mathbb{Z}(N)$.

Proof

(B, τ) : the bottom-top $\mathfrak{g}$ -tangle corr. to M ,

 $(C, \nu) : \quad \text{ } \quad \quad \quad N$

(K, σ) : a surgery presentation of (B, σ) ,

$$(L, v) \quad \simeq \quad (c, v)$$

$M \circ N =$

$v \cup L$
T_g
$\sigma \cup K$

 & surgery along $K \cup \underbrace{\sigma^+ \cup T_g \cup \sigma^- \cup L}_T$.
 

Step 1 $\sigma_{\pm}(K \vee T \vee L) = \sigma_{\pm}(K) + \sigma_{\pm}(L) + g.$

⑤ LK(KUTUL)

$$= \begin{pmatrix} LK(L) & LK(L, v^-) & 0 & 0 \\ LK(v^-, L) & LK(v^-) & -I_g & 0 \\ 0 & -I_g & LK(\tau^+) & LK(\tau^+, K) \\ 0 & 0 & LK(K, \tau^+) & LK(K) \end{pmatrix}$$

Denote

$$P = \begin{pmatrix} I_g & 0 & 0 \\ -Lk(v^-, L) Lk(L)^{-1} & I_g & 0 \\ 0 & 0 & -Lk(\gamma^+, K) Lk(K)^{-1} \\ 0 & 0 & I_k \end{pmatrix}.$$

Then, we have

$$P \cdot Lk(K \cup T \cup L) \cdot P^{-1} = \begin{pmatrix} Lk(L) & 0 & 0 & 0 \\ 0 & X_- & -I_g & 0 \\ 0 & -I_g & X_+ & 0 \\ 0 & 0 & 0 & Lk(K) \end{pmatrix},$$

$$\text{where } X_- = Lk(v^-) - Lk(v^-, L) \cdot Lk(L)^{-1} \cdot Lk(L, v^-),$$

$$X_+ = Lk(\gamma^+) - Lk(\gamma^+, K) \cdot Lk(K)^{-1} \cdot Lk(K, \gamma^+).$$

A homological argument shows $X_+ = Lk_{[-L, 1]_K}(\gamma^+)$,

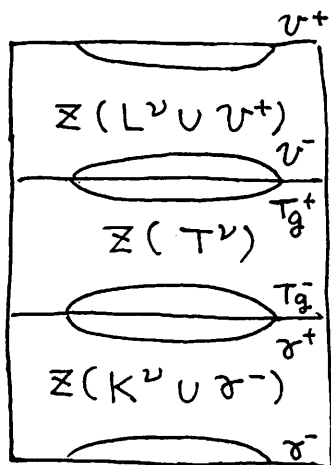
and by Lemma 2.12, $Lk_{[-L, 1]_K}(\gamma^+) = 0$.

Thus, we obtain $\sigma_{\pm}(K \cup T \cup L) = \sigma_{\pm}(K) + \sigma_{\pm}(L) + g$. //

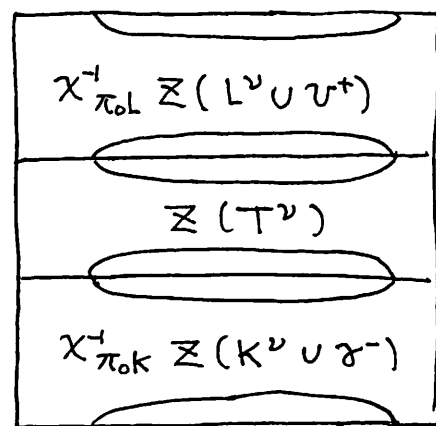
Step 2

$$\begin{aligned} \Sigma(M \circ N) &= \overbrace{U_+^{-\sigma_+(K) - \sigma_+(L) - g}}^{\parallel} \parallel U_-^{-\sigma_-(K \cup T \cup L)} \\ &\parallel \int_{\pi_0(K \cup T \cup L)} \chi_{\pi_0(K \cup T \cup L)}^{-1} \Sigma((K \cup T \cup L)^{\nu_U}(\gamma^- \cup \nu^+)) \\ &\quad \quad \quad (*) \end{aligned}$$

$$(*) = \chi_{\pi_0 T}^{-1} \chi_{\pi_0 K}^{-1} \chi_{\pi_0 L}^{-1}$$



$$= \chi_{\pi_0 T}^{-1}$$



It is known that $\int \pi_0(K \cup T \cup L) = \int \pi_0 T \int \pi_0 K \int \pi_0 L$ ([BGRT2] Prop 2.11) when $Lk_{[-1,1]^3}(K)$ and $Lk_{[-1,1]^3}(L)$ are invertible.

Recall that $Z(B, \sigma) = U_+^{-\sigma_+(L)} \amalg U_-^{-\sigma_-(L)} \amalg \int \pi_0 L \chi_{\pi_0 L}^{-1} Z(L^\nu \cup \sigma)$,
 $Z(C, \nu) = U_+^{-\sigma_+(K)} \amalg U_-^{-\sigma_-(K)} \amalg \int \pi_0 K \chi_{\pi_0 K}^{-1} Z(K^\nu \cup \sigma)$.

Thus, we have

$$Z(M \circ N) = U_+^{-g} \amalg U_-^{-g} \amalg \int \pi_0 T \chi_{\pi_0 T}^{-1}$$

(*)'

$$(*)' = \int_{\{e, f\}} \chi_{\{e, f\}}^{-1} \left\langle \begin{array}{c} a \quad b \\ \begin{array}{|c|c|} \hline \text{ } \\ \hline \end{array} \\ e \end{array} \amalg \begin{array}{c} c \quad d \\ \begin{array}{|c|c|} \hline \text{ } \\ \hline \end{array} \\ f \end{array} \right\rangle$$

$$\begin{array}{c} \begin{array}{|c|c|c|} \hline \chi_{\pi_0 T}^{-1} Z(T^\nu) \\ \hline \end{array} \\ a \quad a \quad a \end{array} \amalg \begin{array}{c} \begin{array}{|c|c|c|} \hline \chi_{\pi_0 T}^{-1} Z(T^\nu) \\ \hline \end{array} \\ b \quad b \quad b \end{array} \amalg \begin{array}{c} \begin{array}{|c|c|c|} \hline \chi_{\pi_0 T}^{-1} Z(T^\nu) \\ \hline \end{array} \\ c \quad c \quad c \end{array} \right\rangle$$

$$= \left\langle \int_{\{e, f\}} \langle \lambda(a, b; e) \amalg \lambda(c, d; f), \chi_{\pi_0 T}^{-1} Z(T^\nu) \rangle_{b, d}, \right.$$

$$\left. \chi_{\pi_0 T}^{-1} Z(T^\nu) \amalg \chi_{\pi_0 T}^{-1} Z(T^\nu) \right\rangle_{a, c}$$

$$\text{Set } \Pi_g = U_+^{-g} \amalg U_-^{-g} \amalg \int_{\{e, f\}} \langle \lambda(a, b; e) \amalg \lambda(c, d; f), \chi_{\pi_0 T}^{-1} Z(T^\nu) \rangle_{b, d}.$$

$$\text{Then, we have } \chi^{-1} Z(M \circ N) = \chi^{-1} Z(M) \circ \Pi_g \circ \chi^{-1} Z(N).$$

Lemma 4.12

M : a Lagrangian g -cobordism from F_g to F_f .

$$\tilde{Z}(M) = \chi^{-1}_{\pi_0} Z(M) \circ \pi_g \in \mathcal{A}(\emptyset, [g]^+ \cup [f]^-)$$

is grouplike and its strut part is $[\frac{Lk(M)}{2}]$.

Proof

(B, τ) : a bottom-top g -tangle in a homology cube corr. to M .

$$\tilde{Z}(M) = \chi^{-1} Z(B, \tau) \circ \pi_g.$$

$\chi^{-1} Z(B, \tau)$ is grouplike by Lemma 3.17, and

π_g is, by Lemma 4.9.

By Lemma 4.5, the composition $\chi^{-1} Z(B, \tau) \circ \pi_g$ is grouplike. //

Thm 4.13

$\tilde{Z}: \mathcal{L}Cob_g \rightarrow {}^{ts}\mathcal{A}$ is a tensor-preserving functor.

Proof

w : non-ass. word of length g .

By Lemma 4.12, $\tilde{Z}$ preserves the composition law.

For $M \in \mathcal{L}Cob_g(u, u')$ and

$$\begin{aligned} \tilde{Z}(M \otimes N) &= \chi^{-1} Z(\overline{[M|N]}) \circ \pi_{g+h} \\ &= \chi^{-1} (Z(M) \otimes Z(N)) \circ \pi_{g+h} \\ &= (\chi^{-1} Z(M) \otimes \chi^{-1} Z(N)) \circ (\pi_g \otimes \pi_h) \\ &= \tilde{Z}(M) \otimes \tilde{Z}(N). \end{aligned}$$

Since $\tilde{Z}(\text{Id}_w)$ is grouplike, we denote $\tilde{Z}(\text{Id}_w) = \left[\sum_{i=1}^g \begin{smallmatrix} i^+ \\ - \end{smallmatrix} \right] \perp \underbrace{\tilde{Z}^Y(\text{Id}_w)}_{Y\text{-part}}.$

We denote $\tilde{Z}(\text{Id}_w) = \emptyset + T + (i\text{-deg} > k).$

Then, by the equation $\tilde{Z}(\text{Id}_w) = \tilde{Z}(\text{Id}_w \circ \text{Id}_w) = \tilde{Z}(\text{Id}_w) \circ \tilde{Z}(\text{Id}_w),$

we have $T = 2T$, i.e. $T = \emptyset$.

Thus, we obtain $\tilde{Z}^Y(\text{Id}_w) = \emptyset.$

§ Mapping class group & LMO functor

The monoid $\text{Cyl}(F_g)$ of homology cylinders

(M, m) is a homology cylinder

$\Leftrightarrow (M, m)$ is a cobordism from F_g to F_g

s.t. $m_{\pm} : H_1(F_g) \rightarrow H_1(M)$ is an isom
and $m_+ = m_-$.

Rem

$I(F_g) := \text{Ker}(\text{MCG}(F_g) \rightarrow \text{Aut } H_1(F_g))$: Torelli group.

$$\begin{aligned} I(F_g) &\hookrightarrow \text{Cyl}(F_g) && \subset \mathcal{L}\text{Cob}(g, g) \\ h &\mapsto (F_g \times [-1, 1], (\text{Id} \times \{-1\}) \cup (h \times \{1\})) \end{aligned}$$

A choice of non-ass word, e.g. $w = (\underbrace{(\dots((\circ \circ) \circ) \dots) \circ}_g)$,

gives a monoid homo. $I(F_g) \xrightarrow{\iota} \mathcal{L}\text{Cob}_g(w, w) \xrightarrow{\sim} {}^{ts}\mathcal{A}(g, g)$.

Lemma

$$M \in \text{Cyl}(F_g) \Leftrightarrow \text{Lk}(M) = \begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix}.$$

Denote

$$\tilde{\mathcal{Z}}^Y : I(F_g) \rightarrow {}^{ts}\mathcal{A}(g, g) \xrightarrow{Y\text{-part}} \mathcal{A}^Y(LgT^+ \cup LgT^-).$$

Rem

$$\begin{aligned} \text{For } D, E \in \text{Cyl}(F_g), \quad \tilde{\mathcal{Z}}(D) &= \left[\begin{array}{c} \vdots^+ \\ \vdots^- \end{array} + \dots + \begin{array}{c} \vdots^{g^+} \\ \vdots^{g^-} \end{array} \right] \perp \tilde{\mathcal{Z}}^Y(D), \\ \tilde{\mathcal{Z}}(E) &= \quad \quad \quad \perp \tilde{\mathcal{Z}}^Y(E). \end{aligned}$$

Then,

$$\begin{aligned} \tilde{\mathcal{Z}}(D \circ E) &= \tilde{\mathcal{Z}}(D) \circ \tilde{\mathcal{Z}}(E) \in {}^{ts}\mathcal{A}(g, g) \\ &= \left[\begin{array}{c} \vdots^+ \\ \vdots^- \end{array} + \dots + \begin{array}{c} \vdots^{g^+} \\ \vdots^{g^-} \end{array} \right] \perp \tilde{\mathcal{Z}}^Y(D) \star \tilde{\mathcal{Z}}^Y(E), \end{aligned}$$

where $x \star y = \langle (x/i^+ \mapsto i^* + i^+), (y/i^- \mapsto i^* + i^-) \rangle_{LgT^*}$.

Example

$$\begin{aligned}
 & \begin{array}{c} 1^- \\ \diagup \quad \diagdown \\ 2^+ \quad 3^- \end{array} \star \begin{array}{c} 1^+ \\ \diagup \quad \diagdown \\ 2^- \quad 3^+ \end{array} = \left\langle \begin{array}{c} 1^- \\ \diagup \quad \diagdown \\ 2^+ \quad 3^- \end{array} + \begin{array}{c} 1^- \\ \diagup \quad \diagdown \\ 2^* \quad 3^- \end{array}, \begin{array}{c} 1^+ \\ \diagup \quad \diagdown \\ 2^- \quad 3^+ \end{array} + \begin{array}{c} 1^+ \\ \diagup \quad \diagdown \\ 2^* \quad 3^+ \end{array} \right\rangle \text{Lg}^* \\
 & = \begin{array}{c} 1^- \quad 1^+ \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ 2^+ \quad 3^- \quad 2^- \quad 3^+ \end{array} + \begin{array}{c} 3^- \quad 1^+ \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ 1^- \quad 3^+ \end{array}.
 \end{aligned}$$

Thm 8.18

$$M \in \text{Cyl}(F_g).$$

If $\tau_n(M)$ is its first nonvanishing Johnson homo.,
 then the tree part of $\tilde{Z}^Y(M)$ is equal to
 $\emptyset + \tau_n(M) + (i\text{-deg} > n).$

Example

$$F_1 = \text{c}(\text{loop}).$$

$$\tilde{Z}^Y(t_c) = \emptyset - \frac{1}{2} \begin{array}{c} 1^+ \\ \diagup \quad \diagdown \\ 1^- \end{array} + \frac{1}{2} \begin{array}{c} 1^+ \quad 1^+ \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ 1^- \quad 1^- \end{array} + (i\text{-deg} \geq 3).$$

$$\frac{1}{2} \begin{array}{c} 1^+ \quad 1^+ \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ 1^- \quad 1^- \end{array} \longleftrightarrow -\frac{1}{2} \begin{array}{c} \beta_1 \quad \beta_1 \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \alpha_1 \quad \alpha_1 \end{array} \longleftrightarrow \tau_2(t_c).$$

Cor 8.22

$\tilde{Z}^Y: \mathcal{I}(F_g) \rightarrow \{\text{units of } (A^Y(\text{Lg}^+ \cup \text{Lg}^-), \star)\}$ is injective.

$$\textcircled{!} \quad \bigcap_{n=0}^{\infty} \underbrace{\pi_1 F_g[n]}_{\substack{\text{lower} \\ \text{central} \\ \text{series}}} = \underbrace{\{1\}}_{\text{known}} \sim \bigcap_{n=0}^{\infty} \text{Ker } \tau_n = \{\text{id}\}. //$$

For $M \in \text{Cyl}(F_g)$ with $w_t(M) = w_b(M) = (\dots((\bullet\bullet)\bullet)\dots\bullet)$,

denote $\bar{M} = \varepsilon^{\otimes g} \circ M \circ \eta^{\otimes g} \in \mathcal{L}\text{Cob}(\emptyset, \emptyset)$

where $\varepsilon = \boxed{\cup}$ $\in \mathcal{L}\text{Cob}(\bullet, \emptyset)$, $\eta = \boxed{\cap}$ $\in \mathcal{L}\text{Cob}(\emptyset, \bullet)$.

Thm 8.24

$$(1) \quad \tilde{\Sigma}(\bar{M}) = (\tilde{\Sigma}^Y(M) / i^+ \mapsto 0, i^- \mapsto 0) \in \mathcal{A}(\emptyset)$$

and $\{\cdot\}$ -coordinates of $\tilde{\Sigma}(\bar{M})$ are equal to $\frac{\lambda(\bar{M})}{2}$ (Casson inv.)

$$(2) \quad \text{For } M, N \in \text{Cyl}(F_g),$$

$$\lambda(\overline{M \circ N}) = \lambda(\bar{M}) + \lambda(\bar{N}) + 2 \text{ (the } \{\cdot\}\text{-coordinate in } \tau_i(M) \star \tau_i(N))$$

Cor 8.6

$M \in \text{Cyl}(F_g)$, s.t.

$$M \stackrel{Y_k}{\sim} (F_g \times [-1, 1], \text{id} \times \{-1\}, \text{id} \times \{1\})$$

Then, $\tilde{\Sigma}^Y(M) = \emptyset + (i\text{-deg} \geq k)$.

Cor

For $\varphi \in \mathcal{I}(F_g)[k]$, $\tilde{\Sigma}^Y(\varphi) = \emptyset + (i\text{-deg} \geq k)$.

Lemma 5.5

$M \in \mathcal{L}\text{Cob}_g(w, v)$ which is presented as follows:

$$M = \begin{array}{|c|} \hline \text{Diagram with } g \text{ crossings} \\ \hline \text{Diagram with } L \text{ crossings} \\ \hline \end{array}$$

where L is a tangle in $[-1, 1]^3$
with $w_b(L) = (v/\bullet \mapsto +)$, $w_t(L) = (w/\bullet \mapsto +)$,

Then,

$$\tilde{\Sigma}(M) = x^{-1} \left(\begin{array}{|c|} \hline \text{Diagram with } g^+ \text{ crossings} \\ \hline \Sigma(L) \\ \hline \end{array} \right).$$

Example



$$\tilde{Z}^Y(t_c) = \emptyset - \frac{1}{2} \begin{array}{c} |^+ \\ \vdots \\ |^- \end{array} + \frac{1}{2} \begin{array}{c} |^+ \\ \vdots \\ |^- \end{array} \begin{array}{c} |^+ \\ \vdots \\ |^- \end{array} + (i\text{-deg} \geq 3).$$

② Step 1

$$\tilde{Z}(t_c) = \tilde{Z} \left(\begin{array}{c} \text{Diagram of a curve with a loop and a vertical line} \end{array} \right)$$

$$= \tilde{Z} \left(\begin{array}{c} \text{Diagram of a curve with a loop and a vertical line, slightly different shape} \end{array} \right)$$

$$= \chi^{-1} \left(\tilde{Z} \left(\begin{array}{c} \text{Diagram of a curve with a loop and a vertical line, further simplified} \end{array} \right) \right)$$

$$= \chi^{-1} \left(\begin{array}{c} \begin{array}{c} |^+ \\ \vdots \\ |^- \end{array} \\ \begin{array}{c} [\frac{1}{2} \vdots] \\ [\frac{1}{2} \vdots] \end{array} \\ \begin{array}{c} [-\frac{1}{2} \vdots] \end{array} \end{array} \right)$$

$$= \chi^{-1} \left(\begin{array}{c} \begin{array}{c} |^+ \\ \vdots \\ |^- \end{array} \\ \begin{array}{c} \text{Diagram of a curve with a loop and a vertical line} \end{array} \end{array} \right)$$

$$\begin{array}{c} \left(\begin{array}{c} \text{Diagram of a curve with a loop and a vertical line} \end{array} \right) \leftarrow \tilde{Z} \left(\begin{array}{c} \text{Diagram of a curve with a loop and a vertical line} \end{array} \right) = \begin{array}{c} \begin{array}{c} |^+ \\ \vdots \\ |^- \end{array} \\ \begin{array}{c} S_2 \Phi \end{array} \end{array} = \begin{array}{c} \begin{array}{c} |^+ \\ \vdots \\ |^- \end{array} \\ \begin{array}{c} S_2 \Phi \\ [-\frac{1}{2} \vdots] \end{array} \end{array} = \tilde{Z} \left(\begin{array}{c} \text{Diagram of a curve with a loop and a vertical line} \end{array} \right) \\ \begin{array}{c} \begin{array}{c} |^+ \\ \vdots \\ |^- \end{array} \\ \begin{array}{c} [-\frac{1}{2} \vdots] \end{array} \end{array} \end{array}$$

Step 2

$$\chi^{-1} \left(\underbrace{\left[\begin{array}{c} \text{[1]} \\ \text{[1]} \end{array} \right]}_{(*)} \right) = \left[\begin{array}{c} \text{[1]} \\ \text{[1]} \end{array} \right] \perp \left[\emptyset - \frac{1}{2} \text{[1]} + \frac{1}{2} \text{[1]} + (i\text{-deg} \geq 3) \right].$$

$$\textcircled{1} (*) = \longrightarrow + \frac{1}{\text{[1]}} + \frac{1}{\text{[1]}} - \frac{1}{\text{[1]}} + \frac{1}{2} \text{[1]} + \frac{1}{6} \text{[1]} + \frac{1}{2} \text{[1]} - \frac{1}{2} \text{[1]} + \frac{1}{2} \text{[1]} + \frac{1}{2} \text{[1]} - \text{[1]} + (\text{deg} \geq 4).$$

$$\textcircled{1} \xrightarrow{\chi^{-1}} \text{[1]},$$

$$\textcircled{2} = \frac{1}{2} \text{[1]} + \text{[1]} = \frac{1}{2} \text{[1]} - \frac{1}{2} \text{[1]} \xrightarrow{\chi^{-1}} \frac{1}{2} \text{[1]} - \frac{1}{2} \text{[1]},$$

$$\begin{aligned} \textcircled{3} &= \frac{1}{6} \text{[1]} + \text{[1]} + \frac{1}{2} \text{[1]} + \frac{1}{2} \left(\text{[1]} + \text{[1]} + \text{[1]} \right) \\ &\quad + \frac{1}{2} \text{[1]} - \left(\text{[1]} + \text{[1]} \right) \\ &= \frac{1}{6} \text{[1]} - \frac{1}{2} \text{[1]} + \frac{1}{2} \text{[1]} + \frac{1}{4} \text{[1]} \\ &\quad \frac{1}{4} \text{[1]} \end{aligned}$$

$$\xrightarrow{\chi^{-1}} \frac{1}{6} \text{[1]} - \frac{1}{2} \left(\text{[1]} + \frac{1}{2} \text{[1]} \right) + \frac{1}{2} \text{[1]} + \frac{1}{4} \text{[1]}.$$

$$\begin{aligned} \therefore \chi^{-1} (*) &= \left[\begin{array}{c} \text{[1]} \\ \text{[1]} \end{array} \right] \perp \left(\emptyset - \frac{1}{2} \text{[1]} + \frac{1}{2} \text{[1]} - \left(-\frac{1}{4} \text{[1]} + \frac{1}{4} \text{[1]} + (\text{deg} \geq 4) \right) \right) \\ &= \left[\begin{array}{c} \text{[1]} \\ \text{[1]} \end{array} \right] \perp \left[\emptyset - \frac{1}{2} \text{[1]} + \frac{1}{2} \text{[1]} + (i\text{-deg} \geq 3) \right]. // \end{aligned}$$